Tableau atoms and a new
Macdonald positivity conjecture

L. Lapointe

Department of Mathematics and Statistics, McGill University
Montréal, Québec, Canada, H3A 2K6
`lapointe@math.mcgill.ca`

A. Lascoux

Institut Gaspard Monge, Université de Marne-la-Vallée
5 Bd Descartes, Champs sur Marne
77454 Marne La Vallée, Cedex, France
`Alain.Lascoux@univ-mlv.fr`

J. Morse

Department of Mathematics, University of Pennsylvania
209 South 33rd Street
Philadelphia, PA 19104, USA
`morsej@math.upenn.edu`

Abstract

Let Λ be the space of symmetric functions and V_k be the subspace spanned by the modified Schur functions $\{S_\lambda[X/(1-t)]\}_{\lambda_1 \leq k}$. We introduce a new family of symmetric polynomials, $\{A_\lambda^{(k)}[X;t]\}_{\lambda_1 \leq k}$, constructed from sums of tableaux using the charge statistic. We conjecture that the polynomials $A_\lambda^{(k)}[X;t]$ form a basis for V_k and that the Macdonald polynomials indexed by partitions whose first part is not larger than k expand positively in terms of our polynomials. A proof of this conjecture would not only imply the Macdonald positivity conjecture, but would substantially refine it. Our construction of the $A_\lambda^{(k)}[X;t]$ relies on the use of tableaux combinatorics and yields various properties and conjectures on the nature of these polynomials. Another important development following from our investigation is that the $A_\lambda^{(k)}[X;t]$ seem to play the same role for V_k as the Schur functions do for Λ. In particular, this has led us to the discovery of many generalizations of properties held by the Schur functions, such as Pieri and Littlewood-Richardson type coefficients.

1

arXiv:math/0008073v2 [math.QA] 17 Nov 2001

1 Introduction

We work with the algebra Λ of symmetric functions in the formal alphabet $x_1, x_2, \ldots$ with coefficients in $\mathbb{Q}(q, t)$. We use λ-ring notation in our presentation and refer those unfamiliar with this device to section 2. It develops that the filtration of Λ given by the spaces

$$V_k = \{S_\lambda[X/(1-t)]\}_{\lambda_1 \leq k}, \quad \text{with } k \in \mathbb{N}, \tag{1.1}$$

provides a natural setting for the study of the q, t-Kostka coefficients, $K_{\lambda\mu}(q, t)$. In fact, this filtration leads to a family of positivity conjectures refining the original Macdonald positivity conjecture, which now holds following the proof [4] of the $n!$ conjecture [2]. To see how this comes about, we first introduce some notation.

We use a modification of the Macdonald integral forms $J_\mu[X; q, t]$ that is obtained by setting

$$H_\mu[X; q, t] = J_\mu[X/(1-t); q, t] = \sum_\lambda K_{\lambda\mu}(q, t) S_\lambda[X]. \tag{1.2}$$

The integral form $J_\mu[X; q, t]$ at $q = 0$ reduces to the Hall-Littlewood polynomial,

$$J_\mu[X; 0, t] = Q_\mu[X; t].$$

We shall also use a modification of $Q_\mu[X; t]$;

$$H_\mu[X; t] = Q_\mu[X/(1-t); t] = \sum_{\lambda \geq \mu} K_{\lambda\mu}(t) S_\lambda[X], \tag{1.3}$$

where $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

This given, we should note that bases for V_k also include the families [12]

$$\{H_\mu[X; t]\}_{\mu_1 \leq k} \quad \text{and} \quad \{H_\mu[X; q, t]\}_{\mu_1 \leq k}. \tag{1.4}$$

Our main contribution is the construction of a new family

$$\{A_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}, \tag{1.5}$$

which we conjecture forms a basis for V_k and whose elements, in a sense that can be made precise, constitute the smallest Schur positive components of V_k. For this reason, we have chosen to call the $A_\lambda^{(k)}[X; t]$ the atoms of V_k.

We begin by outlining the characterization of our atoms which may be compared to the combinatorial construction of the Hall-Littlewood polynomials. The formal sum, or the set, of all semi-standard tableaux (hereafter called tableaux) with evaluation μ will be denoted[1] by $\mathbb{H}_\mu$, with the convention that $\mathbb{H}_0$ is the empty tableau. It was shown in [10] that

$$H_\mu[X; t] = F(\mathbb{H}_\mu) = \sum_{T \in \mathbb{H}_\mu} t^{\text{charge}(T)} S_{\text{shape}(T)}[X], \tag{1.6}$$

---

[1] Double fonts are used to distinguish sets of tableaux or operators on tableaux from functions.

where F is the functional

$$F(T) = t^{\text{charge}(T)} S_{\text{shape}(T)}[X]. \tag{1.7}$$

The formal sum $\mathbb{H}_\mu$ arises from a recursive application of promotion operators $\mathbb{B}_r$ such that $\mathbb{B}_r \mathbb{H}_\lambda = \mathbb{H}_{(r,\lambda)}$:

$$\mathbb{H}_\mu = \mathbb{B}_{\mu_1} \cdots \mathbb{B}_{\mu_{n-1}} \mathbb{B}_{\mu_n} \mathbb{H}_0. \tag{1.8}$$

The operators $\mathbb{B}_r$ are tableau analogues of the operators building recursively the Hall-Littlewood polynomials presented in [5, 3].

To construct the atoms of V_k, we introduce a family of filtering operators, $\mathbb{P}_{\lambda \to k}$, which have the effect of removing certain elements from the sum of tableaux in 1.8 [2]. That is, given a k-bounded partition μ (a partition μ such that $\mu_1 \leq k$), the atom $A_\mu^{(k)}[X;t]$ is

$$A_\mu^{(k)}[X;t] = F\left(\mathbb{A}_\mu^{(k)}\right) = \sum_{T \in \mathbb{A}_\mu^{(k)}} t^{\text{charge}(T)} S_{\text{shape}(T)}[X], \tag{1.9}$$

where $\mathbb{A}_\mu^{(k)}$ is the formal sum of tableaux obtained from

$$\mathbb{A}_\mu^{(k)} = \mathbb{P}_{\mu \to k} \mathbb{B}_{\mu_1} \cdots \mathbb{P}_{(\mu_{n-1},\mu_n) \to k} \mathbb{B}_{\mu_{n-1}} \mathbb{P}_{(\mu_n) \to k} \mathbb{B}_{\mu_n} \mathbb{H}_0. \tag{1.10}$$

Following from this construction is the expansion,

$$A_\mu^{(k)}[X;t] = S_\mu[X] + \sum_{\lambda > \mu} v_{\lambda\mu}^{(k)}(t) S_\lambda[X], \quad \text{with } 0 \subseteq v_{\lambda\mu}^{(k)}(t) \subseteq K_{\lambda\mu}(t), \tag{1.11}$$

where for two polynomials $P, Q \in \mathbb{Z}[q,t]$, we write $P \subseteq Q$ to mean $Q - P \in \mathbb{N}[q,t]$.

Originally, the atoms were empirically constructed by the idea that they could be characterized by 1.11 and the two following properties:

i) for k-bounded partitions λ and μ, and any non-zero coefficient $c(t) \in \mathbb{N}[t]$,

$$A_\lambda^{(k)}[X;t] - c(t) A_\mu^{(k)}[X;t] \neq \sum_\nu v_\nu(t) S_\nu[X], \quad \text{where} \quad v_\nu \in \mathbb{N}[t] \tag{1.12}$$

ii) for any k-bounded partition μ,

$$H_\mu[X;t] = A_\mu^{(k)}[X;t] + \sum_{\lambda > \mu} K_{\lambda\mu}^{(k)}(t) A_\lambda^{(k)}[X;t], \quad \text{with } K_{\lambda\mu}^{(k)}(t) \in \mathbb{N}[t]. \tag{1.13}$$

However, our computer experimentation supported the following stronger conjecture, which connects the atoms to Macdonald polynomials indexed by k-bounded partitions:

$$H_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}^{(k)}(q,t) A_\lambda^{(k)}[X;t], \tag{1.14}$$

---

[2] The effect of the filtering operator is to extract λ-katabolizable tableaux (see Section 2.2 and [18]).

with

$$0 \subseteq K^{(k)}_{\lambda\mu}(q,t) \subseteq K_{\lambda\mu}(q,t) \,. \tag{1.15}$$

This has been the primary motivation for the research that led to this work. In particular, given the positive expansion in 1.11, property 1.14 with 1.15 would not only prove the Macdonald positivity conjecture, but would constitute a substantial strengthening of it.

It will transpire that the atoms are a natural generalization of the Schur functions. In fact, our construction of $A^{(k)}_{\lambda}[X;t]$ yields the property that for large k $(k \geq |\lambda|)$,

$$A^{(k)}_{\lambda}[X;t] = S_{\lambda}[X] \,. \tag{1.16}$$

Thus the atoms of $\Lambda = V_{\infty}$ are none other than the Schur functions themselves. Moreover, computer exploration has revealed that the $A^{(k)}_{\lambda}[X;t]$ have a variety of remarkable properties extending and refining well-known properties of Schur functions. For example, we have observed generalizations of Pieri and Littlewood-Richardson rules, a k-analogue of the Young Lattice induced by the multiplication action of e_1, and a k-analogue of partition conjugation. Further, we have noticed that the atoms satisfy, on any two alphabets X and Y,

$$A^{(k)}_{\lambda}[X+Y;t] = \sum_{|\mu|+|\rho|=|\lambda|} g^{\lambda}_{\mu\rho}(t) \, A^{(k)}_{\mu}[X;t] \, A^{(k)}_{\rho}[Y;t], \quad \text{where } g^{\lambda}_{\mu\rho}(t) \in \mathbb{N}[t] \,.$$

The positivity of the coefficients $g^{\lambda}_{\mu\rho}(t)$ appearing here is a natural property of Schur functions not shared by the Hall-Littlewood or Macdonald functions. Finally, the atoms of V_k, when embedded in the atoms of $V_{k'}$ for $k' > k$, seem to decompose positively:

$$A^{(k)}_{\lambda}[X;t] = A^{(k')}_{\lambda}[X;t] + \sum_{\mu>\lambda} v^{(k\to k')}_{\mu\lambda}(t) \, A^{(k')}_{\mu}[X;t], \quad \text{where } v^{(k\to k')}_{\mu\lambda}(t) \in \mathbb{N}(t) \,. \tag{1.17}$$

The tableaux combinatorics involved in our construction and identity 1.13 suggest that the atoms provide a natural structure on the set of tableaux $\mathbb{H}_{\mu}$. For example, we have observed that for a k-bounded partition μ, $\mathbb{H}_{\mu}$ decomposes into disjoint subsets $\mathbb{A}^{(k)}_T$ indexed by their element of minimal charge. Each of these subsets is characterized by the fact that its cyclage-cocyclage poset structure is isomorphic to that of $\mathbb{A}^{(k)}_{\text{shape}(T)}$, and since

$$\mathcal{F}\left(\mathbb{A}^{(k)}_T\right) = t^{\text{charge}(T)} A^{(k)}_{\text{shape}(T)}[X;t] \,, \tag{1.18}$$

we say that $\mathbb{A}^{(k)}_T$ is a copy of $\mathbb{A}^{(k)}_{\text{shape}(T)}$. Therefore, if $\mathcal{C}^{(k)}_{\mu}$ is the collection of tableaux indexing the copies that occur in the decomposition of $\mathbb{H}_{\mu}$, we have

$$\mathbb{H}_{\mu} = \sum_{T \in \mathcal{C}^{(k)}_{\mu}} \mathbb{A}^{(k)}_T \,. \tag{1.19}$$

Note that the tableaux in $\mathbb{A}_T^{(k)}$ have evaluation μ while those in $\mathbb{A}_{\mathrm{shape}(T)}^{(k)}$ have, from 1.10, evaluation given by the shape of T. Now, 1.18 and 1.19 imply that the coefficients $K_{\lambda\mu}^{(k)}(t)$ occurring in 1.13 are simply given by the formula

$$K_{\lambda\mu}^{(k)}(t) = \sum_{\substack{T \in \mathcal{C}_\mu^{(k)} \\ \mathrm{shape}(T)=\lambda}} t^{\mathrm{charge}(T)} . \tag{1.20}$$

Since the promotion operators $\mathbb{B}_\ell$ acting on $\mathbb{A}_T^{(k)}$ produce collections of tableaux of the same evaluation, we examine their decomposition into copies as well. It appears that

$$\mathbb{B}_\ell \mathbb{A}_T^{(k)} = \sum_{T' \in \mathcal{E}_{T,\ell}^{(k)}} \mathbb{A}_{T'}^{(k)} , \tag{1.21}$$

where $\mathcal{E}_{T,\ell}^{(k)}$ is a suitable subcollection of the tableaux T' of shape ν such that $\nu/\mathrm{shape}(T)$ is a horizontal ℓ-strip. Therefore, formula 1.21 may be considered a refinement of the classical Pieri rules. In fact, letting $t = 1$ and $\mathrm{shape}(T) = \lambda$ in 1.21, we have

$$h_\ell[X]A_\lambda^{(k)}[X;1] = \sum_{\nu \in E_{\lambda,\ell}^{(k)}} A_\nu^{(k)}[X;1], \tag{1.22}$$

where $E_{\lambda,\ell}^{(k)}$ is a subset of the collection of shapes ν such that ν/λ is a horizontal ℓ strip. We shall give a simple combinatorial procedure for determining $E_{\lambda,\ell}^{(k)}$.

When $\ell = 1$ in 1.22, we are led to a k-analogue of the Young lattice. This is the poset whose elements are k-bounded partitions and whose Hasse diagram is obtained by linking an element λ to every $\mu \in E_{\lambda,1}^{(k)}$. In Figure 1, we illustrate the poset obtained for degree 6 with $k = 3$. Moreover, the number of paths in this poset joining the empty partition to the partition λ is simply the number of summands in 1.20 when $\mu = 1^{|\lambda|}$, namely $K_{\lambda,1^{|\lambda|}}^{(k)}(1)$. An analogous observation can be made for a general μ.

Central to our research is the observation that not all of the atoms need to be constructed using 1.10. In fact, for each k there is a distinguished "irreducible" subset of atoms of V_k from which all successive atoms may be constructed simply by applying certain generalized promotion operators. To be more precise, let a partition μ with no more than i parts equal to $k - i$ be called k-irreducible (note, there are $k!$ such partitions). Thus, any k-bounded partition can be obtained by the partition rearrangement of the parts of a k-irreducible partition and a sequence of k-rectangles, partitions of the form $(\ell^{k+1-\ell})$ for $\ell = 1, \ldots, k$. This given, we let the collection of k-irreducible atoms be only the atoms indexed by k-irreducible partitions. This suggests that there are certain generalized promotion operators indexed by k-rectangles yielding that every atom may be written in the form

$$A_\lambda^{(k)}[X;t] = t^c F \left(\mathbb{B}_{R_1} \mathbb{B}_{R_2} \ldots \mathbb{B}_{R_\ell} \mathbb{A}_\mu^{(k)} \right) , \tag{1.23}$$
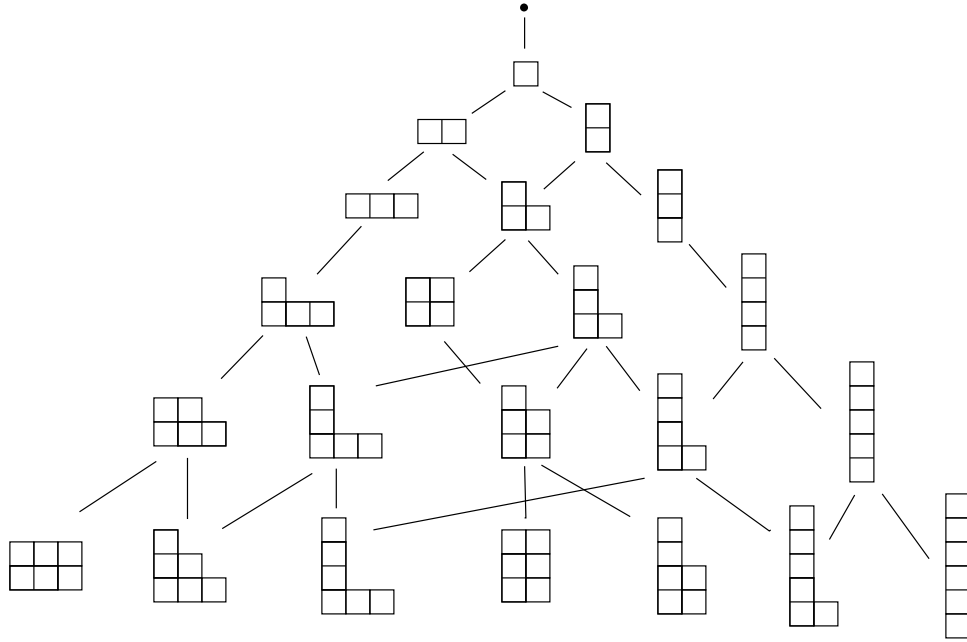
Figure 1: 3-analogue of the Young Lattice

where μ is a k-irreducible partition, $R_1, \ldots, R_\ell$ are certain k-rectangles, and $c \in \mathbb{N}$.

Again we find it interesting to consider the case $t = 1$. First, since the Hall-Littlewood polynomials at $t = 1$ are simply

$$H_{(\mu_1, \ldots, \mu_n)}[X; 1] = h_{\mu_1}[X] \cdots h_{\mu_n}[X], \tag{1.24}$$

we see that V_k reduces to the polynomial ring $V_k(1) = \mathbb{Q}[h_1, \ldots, h_k]$. Further, since the construction in 1.23 is simply multiplication by Schur functions when $t = 1$,

$$A_\lambda^{(k)}[X; 1] = S_{R_1}[X] S_{R_2}[X] \ldots S_{R_\ell}[X] A_\mu^{(k)}[X; 1], \tag{1.25}$$

it thus follows that k-irreducible atoms constitute a natural basis for the quotient $V_k(1)/\mathcal{I}_k$, where $\mathcal{I}_k$ is the ideal generated by Schur functions indexed by k-rectangles. In fact, the irreducible atom basis offers a very beautiful way to carry out operations in this quotient ring: first work in $V_k(1)$ using atoms and then replace by zero all atoms indexed by partitions which are not k-irreducible.

We shall examine our k-analogue of the Young lattice restricted to k-irreducible partitions. Figure 5 gives the case $k = 3$ and $k = 4$, where vertices denote irreducible atoms rather than partitions. Since it can be shown that the collection of monomials of the form $\{h_1^{\epsilon_1}, h_2^{\epsilon_2}, \ldots, h_k^{\epsilon_k}\}_{0 \leq \epsilon_i \leq k-i}$ provides a basis for the quotient $V_k(1)/\mathcal{I}_k$, it follows that the

Hilbert series $F_{V_k(1)/\mathcal{I}_k}(q)$ of this quotient, as well as the rank generating function of the corresponding poset, is given by

$$F_{V_k(1)/\mathcal{I}_k}(q) = \prod_{i=1}^{k-1} \left(1 + q^i + q^{2i} + \cdots + q^{(k-i)i} \right) . \tag{1.26}$$

Finally, we shall make connections between our work and contemporary research in this area. We discovered that tableaux manipulations identical to ours have been used for a different purpose in [16, 17, 18]. In particular, certain cases of the generalized Kostka polynomials can be expressed in our notation as

$$F \left(\mathbb{B}_{R_1} \cdots \mathbb{B}_{R_\ell} \, \mathbb{H}_0 \right) , \tag{1.27}$$

where $R_1, \ldots, R_\ell$ is a sequence of rectangles whose concatenation is a partition [17]. When $R_1, \ldots, R_\ell$ is a sequence of k-rectangles this is simply the case $\mu = \emptyset$ in 1.23. Thus, it is again apparent that an integral part of our work lies in the k-irreducible atoms, without which the atoms in general could not be constructed.

Furthermore, it is known that these generalized Kostka polynomials can be built from the symmetric function operators B_R introduced in [19]. The connection we have made with our atoms and these polynomials thus suggest that any atom can be obtained by applying a succession of operators B_R indexed by k-rectangles to a given irreducible atom $A_\mu^{(k)}[X; t]$:

$$A_\lambda^{(k)}[X; t] = t^c B_{R_1} B_{R_2} \cdots B_{R_\ell} A_\mu^{(k)}[X; t], \quad \text{where } c \in \mathbb{N}. \tag{1.28}$$

Note this is a symmetric function analogue of 1.23 and specializes to 1.25 when $t = 1$.

Contents

2 Background

2.1 Symmetric function theory

Here, symmetric functions are indexed by partitions, or sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$. The order of λ is $|\lambda| = \lambda_1 + \lambda_2 + \dots$, the number of non-zero parts in λ is denoted $\ell(\lambda)$, and $n(\lambda) = \sum_i (i-1)\lambda_i$. We use the dominance order on partitions with $|\lambda| = |\mu|$, where $\lambda \leq \mu$ when $\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i$ for all i. For two partitions λ and μ, $\lambda \cup \mu$ denotes the partition rearrangement of the parts of λ and μ.

Every partition λ may be associated to a Ferrers diagram with λ_i lattice squares in the ith row, from the bottom to top. For example,

$$\lambda = (4,3,1) = \;\blacksquare. \tag{2.1}$$

For each cell $s = (i,j)$ in the diagram of λ, let $\ell'(s), \ell(s), a(s)$, and $a'(s)$ be respectively the number of cells in the diagram of λ to the south, north, east, and west of the cell s. The transposition of a diagram associated to λ with respect to the main diagonal gives the conjugate partition λ'. For example, the conjugate of (4,3,1) is

$$\lambda' = \;\blacksquare\; = (3,2,2,1). \tag{2.2}$$

A skew diagram μ/λ, for any partition μ containing the partition λ, is the diagram obtained by deleting the cells of λ from μ. The thick frames below represent $(5,3,2,1)/(4,2)$.

$$\blacksquare. \tag{2.3}$$

We employ the notation of λ-rings, needing only the formal ring of symmetric functions Λ to act on the ring of rational functions in $x_1, \dots, x_N, q, t$, with coefficients in $\mathbb{Q}$. The action of a power sum p_i on a rational function is, by definition,

$$p_i \left[\frac{\sum_\alpha c_\alpha u_\alpha}{\sum_\beta d_\beta v_\beta} \right] = \frac{\sum_\alpha c_\alpha u_\alpha^i}{\sum_\beta d_\beta v_\beta^i}, \tag{2.4}$$

with $c_\alpha, d_\beta \in \mathbb{Q}$ and u_α, v_β monomials in $x_1, \ldots, x_N, q, t$. Since the ring Λ is generated by power sums, p_i, any symmetric function has a unique expression in terms of p_i, and 2.4 extends to an action of Λ on rational functions. In particular $f[X]$, the action of a symmetric function f on the polynomial $X = x_1 + \cdots + x_N$, is simply $f(x_1, \ldots, x_N)$.

We recall that the Macdonald scalar product, $\langle \ , \ \rangle_{q,t}$, on $\Lambda \otimes \mathbb{Q}(q,t)$ is defined by setting

$$\langle p_\lambda[X], p_\mu[X] \rangle_{q,t} = \delta_{\lambda\mu} \, z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} = \delta_{\lambda\mu} \, z_\lambda \, p_\lambda \left[\frac{1-q}{1-t} \right] , \tag{2.5}$$

where for a partition λ with $m_i(\lambda)$ parts equal to i, we associate the number

$$z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots \tag{2.6}$$

The Macdonald integral forms $J_\lambda[X; q, t]$ are then uniquely characterized [12] by

$$\text{(i)} \ \langle J_\lambda, J_\mu \rangle_{q,t} = 0, \qquad \text{if } \lambda \neq \mu, \tag{2.7}$$

$$\text{(ii)} \ J_\lambda[X; q, t] = \sum_{\mu \leq \lambda} v_{\lambda\mu}(q,t) S_\mu[X], \tag{2.8}$$

$$\text{(iii)} \ v_{\lambda\lambda}(q,t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{\ell(s)+1}), \tag{2.9}$$

where $S_\mu[X]$ is the usual Schur function and $v_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$.

2.2 Tableaux combinatorics

$\mathcal{A}^*$ denotes the free monoid generated by the alphabet $\mathcal{A} = \{1, 2, \ldots\}$ and $\mathbb{Q}[\mathcal{A}^*]$ is the free algebra of $\mathcal{A}$. Elements of $\mathcal{A}^*$ are called words and for E a subset of $\mathcal{A}$, w_E denotes the subword obtained by removing from w all the letters not in E. The degree of a word w is denoted $|w|$ and if w has ρ_1 ones, ρ_2 twos, $\ldots$, and ρ_m m's, then the evaluation of w is $(\rho_1, \ldots, \rho_m)$. For example, $w = 131332$ has degree 6 and evaluation (2,1,3). A word w of degree n is standard iff its evaluation is $(1, \ldots, 1)$. Recall that a word w is Yamanouchi in the letters $a_1 < \ldots < a_h$ if it is such that for every factoring $w = uv$, v contains more a_i than a_j for all $i < j$.

The plactic monoid on the alphabet $\mathcal{A}$ is the quotient $\mathcal{A}^* / \equiv$, where $\equiv$ is the congruence generated by the Knuth relations [6] defined on three letters a, b, c by

$$a \, c \, b \equiv c \, a \, b \qquad (a \leq b < c),$$
$$b \, a \, c \equiv b \, c \, a \qquad (a < b \leq c). \tag{2.10}$$

Two words w and w' are said to be Knuth equivalent iff $w \equiv w'$.

In this paper, a tableau is a filling of a Ferrers diagram with positive integer entries that are nondecreasing in rows and increasing in columns:

$$T = \begin{array}{cccc} \boxed{6} \boxed{7} & & & \\ \boxed{4} \boxed{4} \boxed{5} & & \\ \boxed{1} \boxed{1} \boxed{1} \boxed{2} \boxed{3} \end{array} . \tag{2.11}$$

The word w obtained by reading the entries of a tableau from left to right and top to bottom is said to be a tableau word, or simply a tableau. Our example shows that $w = 6744511123$ is a tableau with evaluation (3,1,1,2,1,1,1). A standard tableau $\mathcal{T}$ is a tableau of evaluation $(1, 1, \ldots, 1)$. For example,

$$\mathcal{T} = \begin{array}{c}\boxed{7}\\\boxed{4}\boxed{6}\\\boxed{1}\boxed{2}\boxed{3}\boxed{5}\end{array}. \tag{2.12}$$

The transpose of a standard tableaux $\mathcal{T}^t$ is defined in the same manner as the transpose of a Ferrers diagram. With $\mathcal{T}$ as given in 2.12, we have

$$\mathcal{T}^t = \begin{array}{c}\boxed{5}\\\boxed{3}\\\boxed{2}\boxed{6}\\\boxed{1}\boxed{4}\boxed{7}\end{array}. \tag{2.13}$$

Since the transpose of a tableau is assured to be a tableau only when the original tableau is standard, this definition is valid only for standard tableaux.

We assume readers are familiar with the Robinson-Schensted correspondence [13, 14],

$$w \longleftrightarrow \big(P(w), Q(w)\big), \tag{2.14}$$

providing a bijection between a word w and a pair of tableaux $\big(P(w), Q(w)\big)$, where $P(w)$ is the only tableau Knuth equivalent to w and $Q(w)$ is a standard tableau.

The ring of symmetric functions is embedded into the plactic algebra by sending the Schur function S_λ to the sum of all tableaux of shape λ [1, 9]. The commutativity of the product $S_\lambda S_\mu \equiv S_\mu S_\lambda$ thus implies bijections among tableaux. In particular, we can define the following action of the symmetric group on words [11]. The elementary transposition σ_i permutes degrees in i and $i+1$. Given a word w of evaluation $(\rho_1, \ldots, \rho_m)$, let u denote the subword in letters $a = i$ and $b = i+1$. The action of the transposition σ_i affects only the subword u and is defined as follows: pair every factor $b\,a$ of u, and let u_1 be the subword of u made out of the unpaired letters. Pair every factor $b\,a$ of u_1, and let u_2 be the subword made out of the unpaired letters. Continue in this fashion as long as possible. When all factors $b\,a$ are paired and unpaired letters of u are of the form $a^r b^s$, σ_i sends $a^r b^s \rightarrow a^s b^r$. For instance, to obtain the action of σ_2 on $w = 123343222423$, we have $u = w_{\{2,3\}} = 233322223$, and the pairings are

$$2\Big(3\big(3(32)2\big)2\Big)23, \tag{2.15}$$

which means that $\sigma_2 u = 233322233$ and $\sigma_2 w = 123343222433$. It is verified in [11] that the σ_i's obey the Coxeter relations and thus provide an action of the symmetric group on words.

We use the notion of charge [9, 10] defined by writing a word w, with evaluation given by a partition, counterclockwise on a circle with a * separating the end of the word from its beginning and then summing the labels that are obtained by the following procedure:

Let $\ell = 0$. Moving clockwise from *, we label with ℓ, the first occurrence of letter 1, then the first occurrence of letter 2 following this 1, then the first occurrence of letter 3 following this 2, etc, with the condition that each time the * is passed, the label ℓ is increased to $\ell + 1$. Once each of the letters $1, 2, 3, \ldots$ have been labeled, we repeat this procedure on the unlabeled letters, again starting at the * with $\ell = 0$. The process ends when all letters have been labeled.

We can define charge on a word w whose evaluation is not a partition by first permuting the evaluation to a partition using σ, and then taking the charge of σw. Figure 2 shows
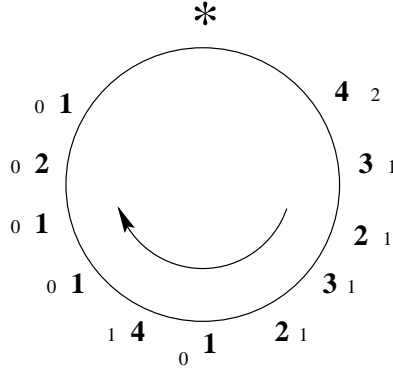

Figure 2: Charge of $w = 12114123234$

that charge$(12114123234) = 0 + 0 + 0 + 0 + 1 + 0 + 1 + 1 + 1 + 1 + 2 = 7$.

The definition of charge gives that a tableau of shape and evaluation μ has charge 0 and thus the combinatorial construction for the Hall-Littlewood polynomials (1.6) implies

$$H_\mu[X;t] = S_\mu[X] + \sum_{\lambda > \mu} K_{\lambda\mu}(t) S_\lambda[X] \,. \tag{2.16}$$

3   Definition of $A_\lambda^{(k)}[X;t]$

Our main contribution is the method for constructing new families of functions whose significance has been outlined in the introduction. The characterization is similar to the combinatorial definition of Hall-Littlewood polynomials using the set of tableaux that arises from a recursive application of promotion operators. Our families also correspond to a set of tableaux generated by the promotion operators, but here we introduce new operators $\mathbb{P}_{\lambda \to k}$ to eliminate undesirable elements. To be precise, we now define the operators involved in our construction.

The promotion operators are defined on a tableau T with evaluation $(\lambda_1, \ldots, \lambda_m)$ by

$$\mathbb{B}_r(T) = \sigma_1 \cdots \sigma_m \, \mathbb{R}_r \, T \,, \tag{3.1}$$

where $\mathbb{R}_r$ adds a horizontal r-strip of the letter $m+1$ to T in all possible ways. For example,

$$\mathbb{B}_3\,\boxed{\begin{smallmatrix}2&2\\1&1\end{smallmatrix}} \;=\; \sigma_1\,\sigma_2\,\mathbb{R}_3\,\boxed{\begin{smallmatrix}2&2\\1&1\end{smallmatrix}} \;=\; \sigma_1\sigma_2\left(\;\boxed{\begin{smallmatrix}2&2\\1&1&3&3&3\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\2&2\\1&1&3&3\end{smallmatrix}}+\boxed{\begin{smallmatrix}3&3\\2&2\\1&1&3\end{smallmatrix}}\;\right) \;=\; \boxed{\begin{smallmatrix}2&2\\1&1&1&3&3\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\2&2\\1&1&1&3\end{smallmatrix}}+\boxed{\begin{smallmatrix}3&3\\2&2\\1&1&1\end{smallmatrix}}\,.$$

Note that the action of σ implies that the resulting tableaux have evaluation $(r,\lambda_1,\dots,\lambda_m)$. While our construction relies on these operators, they generate certain unwanted tableaux. We now present the concepts needed to obtain operators that filter out such elements.

The *main hook-length* of a partition λ, $h_M(\lambda)$, is the hook-length of the cell $s=(1,1)$ in the diagram associated to λ. That is

$$h_M(\lambda) = \ell(s) + a(s) + 1 = \lambda_1 + \lambda_1' - 1 = \lambda_1 + \ell(\lambda) - 1\,. \tag{3.2}$$

For example, if $\lambda = (4,3,1)$, then

$$h_M(\lambda) = \;\begin{array}{c}\text{(diagram)}\end{array}\; = 6\,. \tag{3.3}$$

Any k-bounded partition λ can be associated to a sequence of partitions called the *k-split*, $\lambda^{\to k} = (\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)})$. The k-split of λ is obtained by dividing λ (without changing the order of its entries) into partitions $\lambda^{(i)}$ where $h_M(\lambda^{(i)}) = k$, $\forall\, i \neq r$. For example, $(3,2,2,2,1,1)^{\to 3} = \big((3),(2,2),(2,1),(1)\big)$. Equivalently, we horizontally cut the diagram of λ into partitions $\lambda^{(i)}$ where $h_M(\lambda^{(i)}) = k$. In our example this gives

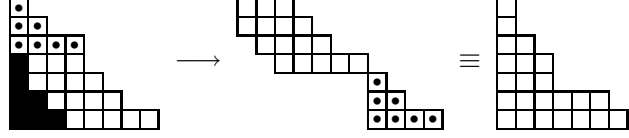$$\begin{array}{c}\text{(diagram)}\end{array} \longrightarrow \begin{array}{c}\text{(diagram)}\end{array}\,. \tag{3.4}$$

Note, the last partition in the sequence $\lambda^{\to k}$ may have main hook-length less than k. As k increases, the k-split of λ will contain fewer partitions. For $k=4$,

$$\begin{array}{c}\text{(diagram)}\end{array} \longrightarrow \begin{array}{c}\text{(diagram)}\end{array} \qquad \text{or} \qquad (3,2,2,2,1,1)^{\to 4} = \big((3,2),(2,2,1),(1)\big)\,. \tag{3.5}$$

When k is big enough $(h_M(\lambda) \leq k)$, then $\lambda^{\to k} = (\lambda)$. i.e., $(3,2,2,2,1,1)^{\to 8} = \big((3,2,2,2,1,1)\big)$.

Let T be a given tableau whose shape contains λ. We shall denote by T_λ, the subtableau of T of shape λ. Let U be the skew tableau obtained by removing T_λ from T, let T_1 be the tableau contained in the first $\ell(\lambda)$ rows of U, and T_2 be the portion of U that is above the $\ell(\lambda)$ rows. Let us denote by T_1T_2 the skew tableau obtained by juxtaposing T_1 to the northwest corner of T_2, and by $\overline{T}$ the unique tableau which is Knuth equivalent to T_1T_2. For instance, in the figure below $\lambda = (3,2,1,1)$, the skew tableau with empty cells is T_1, the
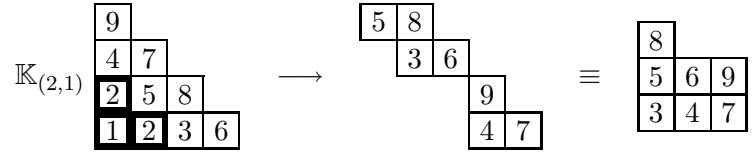
tableau with bullets is T_2, the middle diagram is $T_1 T_2$, and the right diagram is a possible shape for $\overline{T}$.

$$\longrightarrow \qquad \equiv \qquad . \tag{3.6}$$

This construction permits us to define an operation on tableau, $\mathbb{K}_\lambda$, called λ-*katabolism.*

$$\mathbb{K}_\lambda(T) = \begin{cases} \overline{T} & \text{if } \lambda \subseteq \text{shape}(T) \\ 0 & \text{otherwise} \end{cases} .$$

For example, the $(2,1)$-katabolism of $T = 9472581236$ is

$$\mathbb{K}_{(2,1)} \qquad \longrightarrow \qquad \equiv \qquad . \tag{3.7}$$

Note that λ-katabolism was also introduced in [18] and for the case that λ is a row, in [11].
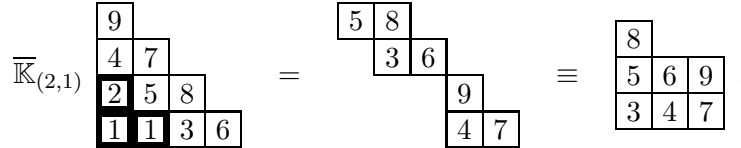
Let $S(\lambda)$ be the set of λ-shaped tableaux with evaluations $(0^m, \lambda_1, \lambda_2, \dots)$, for $m \in \mathbb{N}$. For $\lambda = (3, 2, 2)$, we have

$$S \left(\; \right) = \left\{ \; , \; , \; , \; , \; \dots \right\} . \tag{3.8}$$

This given, the restricted λ-katabolism $\overline{\mathbb{K}}_\lambda$ is defined by setting

$$\overline{\mathbb{K}}_\lambda(T) = \begin{cases} \mathbb{K}_\lambda(T) & \text{if } T_\lambda \in S(\lambda) \\ 0 & \text{otherwise} \end{cases} . \tag{3.9}$$

For example, $\overline{\mathbb{K}}_{(2,1)}$ on the tableau in 3.7 is zero, whereas

$$\overline{\mathbb{K}}_{(2,1)} \qquad = \qquad \equiv \qquad . \tag{3.10}$$

For a sequence of partitions $S = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)})$, we define the filtering operator $\mathbb{P}_S$ using the succession of restricted katabolisms $\overline{\mathbb{K}}_{\lambda^{(\ell)}} \cdots \overline{\mathbb{K}}_{\lambda^{(1)}}$,

$$\mathbb{P}_S(T) = \begin{cases} T & \text{if } \overline{\mathbb{K}}_{\lambda^{(\ell)}} \cdots \overline{\mathbb{K}}_{\lambda^{(1)}}(T) = \mathbb{H}_0 \\ 0 & \text{otherwise} \end{cases} . \tag{3.11}$$

In fact, we only consider the case where S is the sequence of partitions given by $\lambda^{\rightarrow k}$.

Property 1. *The filtering operators* $\mathbb{P}_{\lambda \to k}$ *satisfy the following properties :*

  a. For $T \in S(\lambda)$, *we have* $\mathbb{P}_{\lambda \to k} T = T$ *for all* k *such that* λ *is bounded by* k.

  b. For U *a tableau of* $|\lambda|$ *letters such that* $U \notin S(\lambda)$, $\mathbb{P}_{\lambda \to k} U = 0$ *for all* $k \geq h_M(\lambda)$.

Proof.   By definition 3.11, we must show $\overline{\mathbb{K}}_{\lambda^{(\ell)}} \cdots \overline{\mathbb{K}}_{\lambda^{(1)}} T = \mathbb{H}_0$, for $\lambda^{\to k} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$. Recall that $\overline{\mathbb{K}}_{\lambda^{(1)}} T$ acts by extracting the bottom $\ell(\lambda^{(1)})$ rows of T and inserting into the remainder, any entries not in $T_{\lambda^{(1)}} \in S(\lambda^{(1)})$. Since the bottom $\ell(\lambda^{(1)})$ rows of $T \in S(\lambda)$ are exactly $T_{\lambda^{(1)}} \in S(\lambda^{(1)})$, the katabolism $\overline{\mathbb{K}}_{\lambda^{(1)}} T$ simply removes the bottom $\ell(\lambda^{(1)})$ rows of T. By iteration, we obtain the empty tableau.

For (b), the condition that k is large implies that $\lambda^{\to k} = (\lambda)$. It thus suffices to show that $\overline{\mathbb{K}}_\lambda(U) = 0$. Now $\overline{\mathbb{K}}_\lambda$ acts first by extracting from U, the subtableau $U_\lambda \in S(\lambda)$. If U is of shape λ then $U_\lambda = U \notin S(\lambda)$. If U is not of shape λ, since U has degree $|\lambda|$, then U_λ does not exist. Therefore we have our claim.   □

These filtering operators are those required in the characterization of our families of functions. We thus have the tools to recursively define the central object in our work, the *super atom* of shape λ and level k, $\mathbb{A}_\lambda^{(k)}$.

Definition 2. *Let* $\mathbb{A}_0^{(k)}$ *be the empty tableau. The super atom of a k-bounded partition* λ *is*

$$\mathbb{A}_\lambda^{(k)} = \mathbb{P}_{\lambda \to k} \, \mathbb{B}_{\lambda_1} \left(\mathbb{A}_{(\lambda_2, \lambda_3, \ldots)}^{(k)} \right) . \tag{3.12}$$

For example, given that we know the super atom

$$A_{1,1,1,1}^{(3)} = \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} + \begin{smallmatrix}3\\2\\1\,4\end{smallmatrix} \tag{3.13}$$

we can obtain $\mathbb{A}_{2,1,1,1,1}^{(3)}$ by first acting with the rectangular operator $\mathbb{B}_2$ on 3.13,

$$\mathbb{B}_2 \left(\mathbb{A}_{1,1,1,1}^{(3)} \right) = \mathbb{B}_2 \left(\begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} + \begin{smallmatrix}3\\2\\1\,4\end{smallmatrix} \right) = \begin{smallmatrix}3\\2\\1\,1\,4\,5\end{smallmatrix} + \begin{smallmatrix}3\\2\,5\\1\,1\,4\end{smallmatrix} + \begin{smallmatrix}5\\3\\2\\1\,1\,4\end{smallmatrix} + \begin{smallmatrix}4\\3\\2\,5\\1\,1\end{smallmatrix} + \begin{smallmatrix}4\\3\\2\\1\,1\,5\end{smallmatrix} + \begin{smallmatrix}5\\4\\3\\2\\1\,1\end{smallmatrix} , \tag{3.14}$$

and then to these tableaux, applying the operator $\mathbb{P}_{\lambda \to 3}$, where $\lambda^{\to 3} = \big((2,1), (1,1,1) \big)$;

$$\mathbb{A}_{2,1,1,1,1}^{(3)} = \mathbb{P}_{((2,1),(1,1,1))} \left(\mathbb{B}_2 \, \mathbb{A}_{1,1,1,1}^{(3)} \right) = \begin{smallmatrix}3\\2\,5\\1\,1\,4\end{smallmatrix} + \begin{smallmatrix}4\\3\\2\,5\\1\,1\end{smallmatrix} + \begin{smallmatrix}4\\3\\2\\1\,1\,5\end{smallmatrix} + \begin{smallmatrix}5\\4\\3\\2\\1\,1\end{smallmatrix} . \tag{3.15}$$

Our method for constructing the super atoms allows the derivation of several natural properties. In particular, these properties generally arise as the consequence of those held by the promotion and filtering operators.

Property 3. *For all k-bounded partitions λ, we have*

$$\mathbb{A}_\lambda^{(k)} \subseteq \mathbb{H}_\lambda . \tag{3.16}$$

Proof.   For $\lambda = (\lambda_1, \ldots, \lambda_m)$, recall from 1.8 that

$$\mathbb{H}_\lambda = \mathbb{B}_{\lambda_1} \cdots \mathbb{B}_{\lambda_m} \mathbb{H}_0 . \tag{3.17}$$

On the other hand, following from Definition 2, we have

$$\mathbb{A}_\lambda^{(k)} = \mathbb{P}_{\lambda \to k} \mathbb{B}_{\lambda_1} \cdots \mathbb{P}_{(\lambda_m) \to k} \mathbb{B}_{\lambda_m} \mathbb{A}_0^{(k)} . \tag{3.18}$$

Since $\mathbb{A}_\lambda^{(k)}$ is distinguished from $\mathbb{H}_\lambda$ only by acting with a filtering operator after each application of a $\mathbb{B}_\ell$ operator, we have that every tableau in $\mathbb{A}_\lambda^{(k)}$ is also in $\mathbb{H}_\lambda$. $\qquad\square$

Property 4. *Let T be the tableau of shape and evaluation λ. The super atoms satisfy:*

 i. $T \in \mathbb{A}_\lambda^{(k)}$ for any k such that λ is bounded by k.

 ii. $\mathbb{A}_\lambda^{(k)} = T$ for $k \geq h_M(\lambda)$.

Proof. (i): Recall that $\mathbb{A}_\lambda^{(k)} = \mathbb{P}_{\lambda \to k} \mathbb{B}_{\lambda_1} \mathbb{A}_{\lambda_2, \ldots, \lambda_\ell}^{(k)}$. Assume by induction that $U \in \mathbb{A}_{\lambda_2, \ldots, \lambda_\ell}^{(k)}$ where U has shape and evaluation $(\lambda_2, \ldots, \lambda_\ell)$. $\mathbb{R}_{\lambda_1} U$ produces a sum of tableaux, one being the tableau of shape $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ which is then sent to the tableau T of shape and evaluation λ under the action of the symmetric group. It thus suffices to show that T is not eliminated by $\mathbb{P}_{\lambda \to k}$ for all k. This is shown in Property 1(a).

 (ii): In particular, (i) implies that $T \in \mathbb{A}_\lambda^{(k)}$ for $k \geq h_M(\lambda)$. By the definition of $\mathbb{A}_\lambda^{(k)}$, it thus suffices to show that $\mathbb{P}_{\lambda \to k} U = 0$ for all $U \neq T$. This is true by Property 1(b). $\qquad\square$

As with the definition of the Hall-Littlewood polynomials, we associate symmetric functions to our super atoms.

Definition 5. *With F as in 1.7, we define the symmetric function atoms by*

$$A_\lambda^{(k)}[X; t] = F\left(\mathbb{A}_\lambda^{(k)}\right) . \tag{3.19}$$

Properties we have given for the super atoms allow us to deduce several properties of these functions. For example, an immediate consequence of Property 4(ii) is

Property 6. *When k is large ($k \geq h_M(\lambda)$), we have*

$$A_\lambda^{(k)}[X; t] = S_\lambda[X] . \tag{3.20}$$

Property 7. *The atoms are linearly independent and have an expansion of the form*

$$A_\lambda^{(k)}[X;t] = S_\lambda[X] + \sum_{\mu > \lambda} v_{\mu\lambda}^{(k)}(t)\, S_\mu[X]\,, \quad \text{where } v_{\mu\lambda}^{(k)}(t) \in \mathbb{N}[t]\,. \tag{3.21}$$

Proof.   We have shown that $\mathbb{A}_\lambda^{(k)} \subseteq \mathbb{H}_\lambda$. Thus by Definition 5, the triangularity of $A_\lambda^{(k)}[X;t]$ follows from the triangularity of $H_\lambda[X;t]$ (see 2.16). Further, Property 4 implies that the tableau T of shape and evaluation λ occurs in $\mathbb{A}_\lambda^{(k)}$ and therefore, $F(T) = S_\lambda[X]$ occurs in $A_\lambda^{(k)}$ (T has charge zero).   □

4   Main conjecture

Our work to characterize the atoms was originally motivated by the belief that these polynomials play an important role in understanding the q,t-Kostka coefficients. More precisely,

Conjecture 8. *For any partition λ bounded by k,*

$$H_\lambda[X;q,t] = \sum_{\mu;\mu_1 \leq k} K_{\mu\lambda}^{(k)}(q,t)\, A_\mu^{(k)}[X;t]\,, \quad \text{where } K_{\mu\lambda}^{(k)}(q,t) \in \mathbb{N}[q,t]\,. \tag{4.1}$$

For example, we have

$$
\begin{aligned}
H_{2,1,1}[X;q,t] &= t\, A_{2,2}^{(2)}[X;t] + (1 + qt^2)\, A_{2,1,1}^{(2)}[X;t] + q\, A_{1,1,1,1}^{(2)}[X;t] \\
&= t^2\, A_{3,1}^{(3)}[X;t] + (t + qt^2)\, A_{2,2}^{(3)}[X;t] + (1 + qt^2) A_{2,1,1}^{(3)}[X;t] + q\, A_{1,1,1,1}^{(3)}[X;t] \\
&= t^3\, A_4^{(k \geq 4)}[X;t] + (t + t^2 + qt^3)\, A_{3,1}^{(k \geq 4)}[X;t] + (t + qt^2)\, A_{2,2}^{(k \geq 4)}[X;t] \\
&\quad + (1 + qt + qt^2) A_{2,1,1}^{(k \geq 4)}[X;t] + q\, A_{1,1,1,1}^{(k \geq 4)}[X;t]\,.
\end{aligned}
\tag{4.2}
$$

This conjecture implies that the atoms of level k form a basis for V_k. Further, since the atoms expand positively in terms of Schur functions 3.21, our conjecture also implies Macdonald's positivity conjecture on the $H_\lambda[X;q,t]$ in V_k. Since Property 6 gives

$$K_{\mu\lambda}^{(k)}(q,t) = K_{\mu\lambda}(q,t) \quad \text{for } k \geq |\lambda|\,, \tag{4.3}$$

we see that this conjecture is a generalization of Macdonald's conjecture.

    In fact, our conjecture refines the original Macdonald conjecture in the following sense: substituting 3.21, the positive Schur function expansion of atoms, into 4.1, we have

$$H_\lambda[X;q,t] = \sum_\mu K_{\mu\lambda}^{(k)}(q,t) \sum_{\nu \geq \mu} v_{\nu\mu}^{(k)}(t)\, S_\nu[X]\,. \tag{4.4}$$

On the other hand, since the q, t-Kostka coefficients appear in the expansion

$$H_\lambda[X; q, t] = \sum_\nu K_{\nu\lambda}(q, t) \, S_\nu[X] \,, \tag{4.5}$$

we have that

$$\begin{aligned} K_{\nu\lambda}(q, t) &= \sum_{\mu \leq \nu} K_{\mu\lambda}^{(k)}(q, t) \, v_{\nu\mu}^{(k)}(t) \\ &= K_{\nu\lambda}^{(k)}(q, t) + \sum_{\mu < \nu} K_{\mu\lambda}^{(k)}(q, t) \, v_{\nu\mu}^{(k)}(t) \,. \end{aligned} \tag{4.6}$$

Since $v_{\nu\mu}^{(k)}(t)$ is in $\mathbb{N}[q, t]$, Conjecture 8 implies that

$$K_{\mu\lambda}^{(k)}(1, 1) \leq K_{\mu\lambda}(1, 1) \,, \tag{4.7}$$

where $K_{\mu\lambda}(1, 1)$ is known to be the number of standard tableaux of shape μ. Thus, the problem of finding a combinatorial interpretation for the $K_{\mu\lambda}(q, t)$ coefficients (i.e. associating statistics to standard tableaux) is reduced to obtaining statistics for the fewer $K_{\mu\lambda}^{(k)}(q, t)$.

Based on our conjecture, we have the following corollary concerning the expansion of Hall-Littlewood polynomials in terms of our atoms.

Corollary 9. *Assuming Conjecture 8 holds, we have, for any partition λ bounded by k,*

$$H_\lambda[X; t] = \sum_{\mu \geq \lambda} K_{\mu\lambda}^{(k)}(t) \, A_\mu^{(k)}[X; t] \quad \text{where} \quad K_{\mu\lambda}^{(k)}(t) \in \mathbb{N}[t] \,. \tag{4.8}$$

If we consider this corollary as the result of applying F to an identity on tableaux,

$$F\left(\mathbb{H}_\lambda\right) = \sum_\mu K_{\mu\lambda}^{(k)}(t) \, F\left(\mathbb{A}_\mu^{(k)}\right) \,, \quad \text{where} \quad K_{\mu\lambda}(t) \in \mathbb{N}[t] \,, \tag{4.9}$$

then it suggests that the set of all tableaux with evaluation λ can naturally be decomposed into subsets that are mapped under F to the atoms $A_\mu^{(k)}[X; t]$. Here, $K_{\mu\lambda}^{(k)}(1)$ corresponds to the number of times such a subset occurs in $\mathbb{H}_\lambda$ which, by 4.6, is such that

$$K_{\mu\lambda}^{(k)}(1) \leq K_{\mu\lambda}(1) \,, \tag{4.10}$$

where $K_{\mu\lambda}(1)$ is the number of tableaux with evaluation λ and shape μ. These subsets will be called *copies* of $\mathbb{A}_\mu^{(k)}$ and they will provide a natural decomposition for the set of tableaux of a given evaluation.

5   Embedded tableaux decomposition

We expect from 4.9 that the set of all tableaux with given evaluation can be decomposed into subsets associated to our super atoms. These subsets will be characterized by a cyclage-cocyclage ranked-poset structure [11].

For tableau $T = xw$ where x is not the smallest letter of T, we define T' to be the unique tableau such that $T' \equiv wx$. The mapping $T \to T'$ is a called a *cyclage* and is such that $\mathrm{charge}(T') = \mathrm{charge}(T) + 1$ if the evaluation of T is a partition. For tableau $T = wx$ where x is not the smallest letter of T, we define T' to be the unique tableau such that $T' \equiv xw$. The *cocyclage* is the mapping $T \to T'$ and is such that $\mathrm{charge}(T') = \mathrm{charge}(T) - 1$ if the evaluation of T is a partition.

On any collection of tableaux $\mathbb{T}$ of the same evaluation, we can define a poset $(\mathbb{T}, <_{cc})$. In the case where the evaluation is a partition, the poset is defined by linking any two tableaux T and T' if T is obtained from T', or vice versa, using either a cyclage or a cocyclage. In the case where the evaluation is not a partition, we first permute the evaluation to a partition by using an element σ of the symmetric group, and then construct the poset by linking any two tableaux T and T' if σT is obtained from $\sigma T'$, or vice versa, using either a cyclage or a cocyclage. This induces a partial order on $\mathbb{T}$, such that if $T <_{cc} T'$ then $\mathrm{charge}(T) < \mathrm{charge}(T')$. For example, the poset $\left(\mathbb{A}^{(4)}_{3,2,2,1,1}, <_{cc} \right)$ is

$$
\tag{5.1}
$$
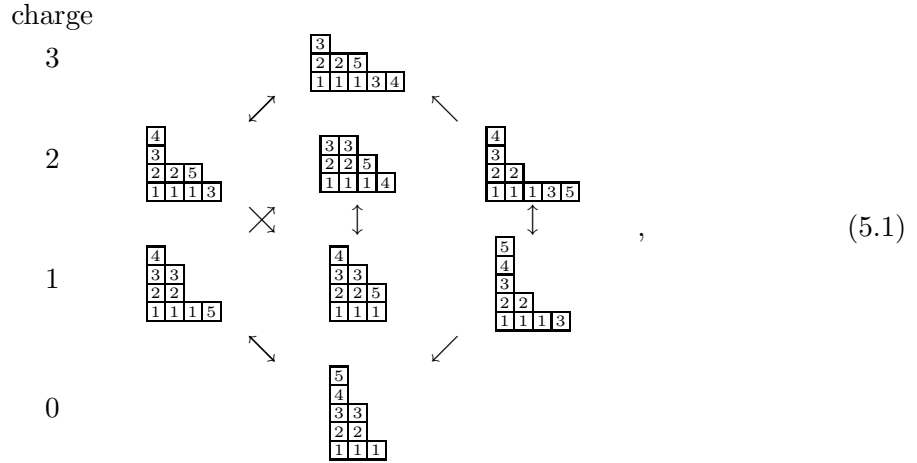
where the arrows indicate the cyclage and cocyclage relations between tableaux.

Conjecture 10. *The cyclage and cocyclage induce a <u>connected</u> ranked-poset structure on the set of tableaux contained in a given super atom $\mathbb{A}^{(k)}_{\lambda}$.*

Given a collection of tableaux $\mathbb{T}$, the Hasse diagram of the poset $(\mathbb{T}, <_{cc})$ with vertices labeled by shapes of the corresponding tableaux will be denoted $\Gamma_{\mathbb{T}}$. We use the symbol

$\Gamma_\lambda^{(k)}$ when $\mathbb{T} = \mathbb{A}_\lambda^{(k)}$. For example, the Hasse diagram $\Gamma_{3,2,2,1,1}^{(4)}$, associated to $\mathbb{A}_{3,2,2,1,1}^{(4)}$, is

charge

$$(5.2)$$
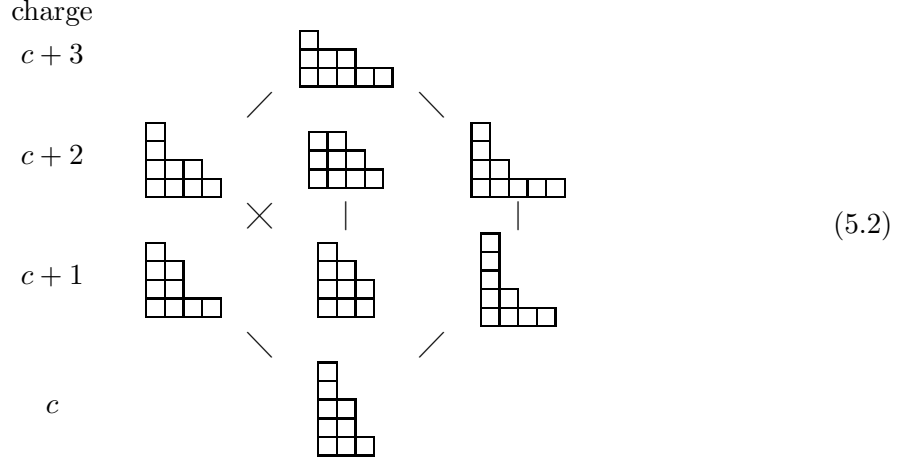
We can now define the subsets associated to our atoms.

Definition 11. *If a set of tableaux $\mathbb{T}$ has the properties:*

    1. *T is the tableau of minimal charge in $\mathbb{T}$*

    2. *$\Gamma_\mathbb{T} = \Gamma_{shape(T)}^{(k)}$,*   (5.3)

then this set is called a copy of the atom $\mathbb{A}_{shape(T)}^{(k)}$ and is denoted $\mathbb{A}_T^{(k)}$.

This given, if the posets are connected (Conjecture 10) then the charges associated to the elements of a super atom $\mathbb{A}_\lambda^{(k)}$ differ from those of a copy atom $\mathbb{A}_T^{(k)}$ by a common factor. Furthermore, since there is a unique element of zero charge in $\mathbb{A}_\lambda^{(k)}$ (the tableau with shape and evaluation λ), then it is the minimal element in $\mathbb{A}_T^{(k)}$ and we have

$$F\left(\mathbb{A}_T^{(k)}\right) = t^{\text{charge}(T)} A_\lambda^{(k)}[X;t], \quad \text{where } shape(T) = \lambda. \qquad (5.4)$$

Note also that the tableaux in $\mathbb{A}_\lambda^{(k)}$ have evaluation λ while those in $\mathbb{A}_T^{(k)}$ have evaluation given by the evaluation of T. That is, $\mathbb{A}_\lambda^{(k)} = \mathbb{A}_T^{(k)}$ only if T is of shape and evaluation λ.

The copies of $\mathbb{A}^{(4)}_{3,2,2,1,1}$ include $\mathbb{A}^{(4)}_{863925147}$, given by

$$
\begin{array}{ll}
\text{charge} & \\
13 & \\
12 & \\
 & \hspace{10cm}(5.5)\\
11 & \\
10 &
\end{array}
$$
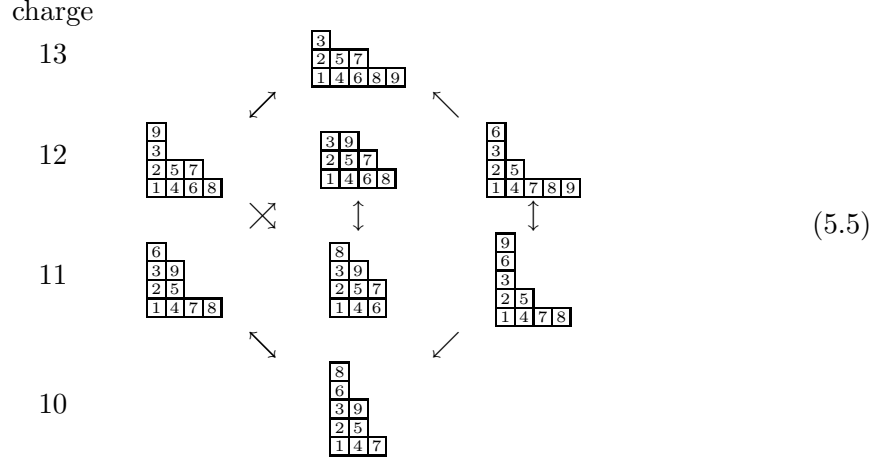
It appears that there is a unique way to decompose the set of all tableaux $\mathbb{H}_\mu$ into atoms of level $k \geq \mu_1$. More precisely, we let $\mathcal{C}^{(k)}_\mu$ denote the collection of all tableaux T with evaluation μ where $\mathbb{A}^{(k)}_T$ is a copy of a super atom of level k. Then

Conjecture 12. *For any partition μ bounded by k, we have*

$$
\mathbb{H}_\mu = \sum_{T \in \mathcal{C}^{(k)}_\mu} \mathbb{A}^{(k)}_T . \tag{5.6}
$$

From Corollary 9 and 5.4, an implication of this identity under the mapping F is:

Corollary 13. *The k-Kostka-Foulkes polynomials are simply*

$$
K^{(k)}_{\lambda\mu}(t) = \sum_{\substack{T \in \mathcal{C}^{(k)}_\mu \\ shape(T)=\lambda}} t^{charge(T)} . \tag{5.7}
$$

One method to obtain the set $\mathcal{C}^{(k)}_\mu$ is as follows: The element of minimal charge in $\mathbb{H}_\mu$ has shape μ and is thus also the minimal element of $\mathbb{A}^{(k)}_\mu$. Remove from $\mathbb{H}_\mu$, all tableaux in $\mathbb{A}^{(k)}_\mu$. Choose a tableau T with minimal charge from those that remain. T must index a copy of the atom $\mathbb{A}^{(k)}_\lambda$ where $\lambda = \text{shape}(T)$. From the Hasse diagram $\Gamma^{(k)}_\lambda$, it is possible to find and remove all tableaux in the atom $\mathbb{A}^{(k)}_T$. Repeat this procedure always on an element of minimal charge in the resulting sets. The collection of these minimal elements is $\mathcal{C}^{(k)}_\mu$.

Evidence suggests that this method also provides a direct decomposition of any copy atom of a level k into copy atoms of level $k' > k$.

Conjecture 14. *For any atom* $\mathbb{A}_T^{(k)}$ *such that* $\mathrm{shape}(T)$ *is bounded by* k, *and any* $k' > k$,

$$\mathbb{A}_T^{(k)} = \sum_{T' \in \mathcal{D}_T^{(k \to k')}} \mathbb{A}_{T'}^{(k')}, \tag{5.8}$$

for some collection of tableaux $\mathcal{D}_T^{(k \to k')}$.

On the level of functions, this translates into

Corollary 15. *If* λ *is a partition bounded by* k, *and* $k' > k$, *then*

$$A_\lambda^{(k)}[X;t] = A_\lambda^{(k')}[X;t] + \sum_{\mu > \lambda} v_{\mu\lambda}^{(k \to k')}(t)\, A_\mu^{(k')}[X;t], \quad \text{where } v_{\mu\lambda}^{(k \to k')}(t) \in \mathbb{N}[t]. \tag{5.9}$$

This conjecture is a generalization of the result presented in Property 7 since we recover $v_{\mu\lambda}^{(k \to k')}(t) = v_{\mu\lambda}(t)$ when $k' \geq |\lambda|$.

For examples that support the preceding conjectures, refer to Figures 6 and 7. These figures also suggest that the number of elements in an atom, at increasing charges, forms a unimodal sequence. Since an atom has a unique minimal element, these sequences always start with 1.

Conjecture 16. *Given any atom*

$$A_\lambda^{(k)}[X;t] = \sum_{\mu \geq \lambda} v_{\mu\lambda}^{(k)}(t)\, S_\mu[X], \tag{5.10}$$

the numbers

$$\#_i = \sum_{\mu \geq \lambda} v_{\mu\lambda}^{(k)}(t)\Big|_{t^i}, \tag{5.11}$$

are such that $[\#_0, \#_1, \dots]$ *is a unimodal sequence.*

For example, the unimodal sequence associated to $A_{3,2,2,1,1,1}^{(4)}[X;t]$ is $[1,3,5,5,3,1]$:

$$
\begin{aligned}
A_{3,2,2,1,1,1}^{(4)}[X;t] ={}& S_{3,2,2,1,1,1} + t\, S_{4,2,1,1,1,1} + t\, S_{3,3,2,1,1} + \left(t + t^2\right) S_{4,2,2,1,1} + t^2\, S_{3,3,3,1} \\
& + t^2\, S_{4,3,1,1,1} + \left(t^2 + t^3\right) S_{5,2,1,1,1} + \left(t^2 + t^3\right) S_{4,3,2,1} + t^3\, S_{5,2,2,1} \\
& + t^3\, S_{4,3,3} + \left(t^3 + t^4\right) S_{5,3,1,1} + t^4\, S_{6,2,1,1} + t^4\, S_{5,3,2} + t^5\, S_{6,3,1}\,.
\end{aligned}
\tag{5.12}
$$

We will see later (Corollary 37) that these sequences also end with a one, that is, an atom has a unique element of maximal charge. We will also provide a way to obtain the shape of this maximal element. Note that the sequences are not necessarily symmetric. For instance, from Figure 7 we see that the sequence associated to $A_{1,1,1,1,1}^{(2)}[X;t]$ is [1,1,2,2,1].

We finish this section by stating a conjecture that reiterates the importance of the atoms as a natural basis for V_k.

Conjecture 17. *For any two alphabets X and Y,*

$$A_\lambda^{(k)}[X + Y; t] = \sum_{|\mu|+|\rho|=|\lambda|} g_{\mu\rho}^\lambda(t)\, A_\mu^{(k)}[X; t]\, A_\rho^{(k)}[Y; t]\,, \qquad (5.13)$$

with $g_{\mu\rho}^\lambda(t) \in \mathbb{N}[t]$.

It is important to note that the positivity of the coefficients $g_{\mu\rho}^\lambda(t)$ appearing here is a natural property of Schur functions that is not shared by the Hall-Littlewood or Macdonald functions.

6   Irreducible atoms

We have now seen that the super atoms can be constructed by generating sets of tableaux with promotion operators $\mathbb{B}_r$ and then eliminating undesirable elements using the projection operators $\mathbb{P}_{\lambda \to k}$. Further, we have given a method to obtain copies of the super atoms allowing us to decompose the set of all tableaux with a given evaluation and to provide natural properties on the functions $A_\lambda^{(k)}[X; t]$.

Remarkably, it appears that there is a method to construct many of the atoms without generating any undesirable elements. In fact, what could be seen as the 'DNA' of our atoms is a subset of irreducible atoms for each V_k, from which all successive atoms of V_k may be obtained by simply applying a generalized version of the promotion operators.

To be more precise, let a rectangular partition of the form $(\ell^{k+1-\ell})$ be referred to as a k-rectangle and a partition with no more than i parts equal to $k - i$ be called k-irreducible.

Definition 18. *The collection of k-irreducible atoms is composed of atoms indexed by k-irreducible partitions. If an atom is not irreducible, then it is said to be reducible.*

Property 19. *There are $k!$ distinct k-irreducible partitions.*

Proof.   A partition λ is k-irreducible if and only if λ has no more than i parts equal to $k - i$. There are obviously $k!$ such partitions.   □

The irreducible atoms of level 1,2 and 3 are

$$
\begin{aligned}
k = 1 : \quad & A_0^{(1)}\,, \\
k = 2 : \quad & A_0^{(2)}\,, \quad A_1^{(2)}\,, \\
k = 3 : \quad & A_0^{(3)}\,, \quad A_1^{(3)}\,, \quad A_2^{(3)}\,, \quad A_{1,1}^{(3)}\,, \quad A_{2,1}^{(3)}\,, \quad A_{2,1,1}^{(3)}\,.
\end{aligned}
\qquad (6.1)
$$

Any k-bounded partition μ is of the form $\mu = \lambda \cup R_1 \cup \cdots \cup R_n$, where λ is a k-irreducible partition and $R_1, \ldots, R_n$ is a sequence of k-rectangles. In fact, any atom of level k can be

obtained from a k-irreducible atom by the application of certain generalized promotion operators that are indexed by k-rectangles.

Before we can introduce these promotion operators, we need to define an operation which generalizes σ_i. We define $\sigma_i^{(h)}$ to send a word w of evaluation $(\rho_1, \rho_2, \dots)$ to a word w' of evaluation $(\rho_1, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_{i+h}, \rho_i, \rho_{i+h+1}, \dots)$. The operation $\sigma_i^{(h)}$ only acts on the subword $w_{\{i,\dots,i+h\}}$, and can thus be defined in generality from the special case $i = 1$. Let w be a word in $1, \dots, h+1$ and let $\big(P(w), Q(w)\big)$ denote the pair of tableaux in RS-correspondence with w (see 2.14). If w'' is the word obtained from w by first erasing all occurrences of the letter 1 and then decreasing the remaining letters by 1, then the shape of $P(w'')$ differs from that of $P(w)$ by a horizontal strip. Let T' be the tableau obtained from $P(w'')$ by filling the horizontal strip with $(h+1)$'s, and let w' be the word which is in RS-correspondence with the pairs of tableaux $\big(P(w'), Q(w')\big) = \big(T', Q(w)\big)$. We now define $\sigma_1^{(h)}$ by setting

$$\sigma_1^{(h)}(w) = w'. \tag{6.2}$$

It can be shown that $\sigma_i^{(1)} = \sigma_i$, and thus $\sigma_i^{(h)}$ generalizes σ_i. Note that $\sigma_i^{(h)}$ happens to be a special case of an operation defined in [16].

The rectangular promotion operators are defined in a manner similar to the promotion operators. That is, on a tableau T of evaluation $(\lambda_1, \dots, \lambda_m)$,

$$\mathbb{B}_{(\ell^h)}(T) = \sigma_1^{(h)} \cdots \sigma_m^{(h)} \, \mathbb{R}_{(\ell^h)} \, T \tag{6.3}$$

generates a sum of tableaux with evaluation $(\ell^h, \lambda_1, \dots, \lambda_m)$ by applying a rectangular analogue of $\mathbb{R}_r$. This operator, $\mathbb{R}_{(\ell^h)}$, acts by adding to T, a horizontal ℓ-strip of the letter $m+1$, a horizontal ℓ-strip of $m+2$, $\dots$, and a horizontal ℓ-strip of $m+h$ in all possible ways such that the tableaux are Yamanouchi in the added letters [3]. Since $\sigma_i^{(1)} = \sigma_i$ for $h = 1$, we recover the previously defined promotion operator $\mathbb{B}_\ell$.

$$\mathbb{B}_{(2^3)} \,\begin{smallmatrix} 2 \\ 1\,2 \end{smallmatrix} = \sigma_1^{(3)} \sigma_2^{(3)} \, \mathbb{R}_{(2^3)} \,\begin{smallmatrix} 2 \\ 1\,2 \end{smallmatrix} = \sigma_1^{(3)} \sigma_2^{(3)} \left(\; \raisebox{-1em}{\tiny tableaux} \; + \; \cdots \; \right)$$

$$= \; \raisebox{-1em}{\tiny tableaux} \; + \; \cdots \; . \tag{6.4}$$

In fact, it seems that the inverse of the $\mathbb{B}_{(\ell^h)}$ action on an atom of level k is simply rectangular-katabolism,

---

[3] This is the multiplication involved in computing the Littlewood-Richardson coefficients in the product of a Schur function of the shape of T by a Schur function indexed by a rectangular partition.

Conjecture 20. *If τ is a translation of the letters in $\mathbb{A}_T^{(k)}$, we have*

$$\mathbb{K}_{(\ell^{k-\ell+1})}\,\mathbb{B}_{(\ell^{k-\ell+1})}\,\mathbb{A}_T^{(k)} \;=\; \tau\mathbb{A}_T^{(k)}\,. \qquad (6.5)$$

The tableaux in 6.4 are sent, under $\mathbb{K}_{(2^3)}$, to $\boxed{\begin{smallmatrix}5\\4\,5\end{smallmatrix}}$ (a translation of the atom $\mathbb{A}_{212}^{(3)}$). Note that in general $\mathbb{K}_{(\ell^h)}\,\mathbb{B}_{(\ell^h)}\,T$ is not necessarily equal to a translation of the letters in T.

This conjecture supports the very important idea that any atom can be obtained from an irreducible atom simply by applying a sequence of rectangular promotion operators. That is,

Conjecture 21. *The operator $\mathbb{B}_{(\ell^{k-\ell+1})}$ acts on any copy $\mathbb{A}_T^{(k)}$ of $\mathbb{A}_\lambda^{(k)}$ by*

$$\mathbb{B}_{(\ell^{k-\ell+1})}\,\mathbb{A}_T^{(k)} = \mathbb{A}_{T'}^{(k)}\,, \qquad (6.6)$$

for a tableau T' of shape $\lambda \cup (\ell^{k-\ell+1})$.

For instance, by applying $\mathbb{B}_{(3)}$ to $\mathbb{A}_{213}^{(3)} = \boxed{\begin{smallmatrix}2\\1\,3\end{smallmatrix}}$, we obtain a copy of $\mathbb{A}_{3,2,1}^{(3)}$:

$$\mathbb{B}_{(3)}\,\boxed{\begin{smallmatrix}2\\1\,3\end{smallmatrix}} = \boxed{\begin{smallmatrix}3\\2\,4\\1\,1\,1\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\1\,1\,1\,3\,4\end{smallmatrix}} + \boxed{\begin{smallmatrix}4\\2\\1\,1\,1\,3\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\,4\\1\,1\,1\,3\end{smallmatrix}} = \mathbb{A}_{324111}^{(3)}\,. \qquad (6.7)$$

Conjecture 21 not only reveals the importance of the set of irreducibles, but also provides a convenient way to obtain copies using a simple transformation on tableau. Given a tableau, the transformation $\mathbb{L}_T$ is defined by replacing the shape(T)-subtableau with T and then adjusting the remaining entries to start with $s + 1$, where s is the largest letter of T. $\mathbb{L}_T$ satisfies several properties on the set of tableaux, denoted $\mathbb{H}_{\mu|_{(\ell^h)}}$, with evaluation μ (not necessarily a partition) whose restriction to the h smallest letters gives exactly the subtableau of shape and evaluation (ℓ^h).

Property 22. *Let $\mathbb{T} \subseteq \mathbb{H}_{\mu|_{(\ell^h)}}$ be a set containing a unique element of minimal charge and whose poset $(\mathbb{T}, <_{cc})$ is connected. For any tableau T of shape (ℓ^h), $\overline{\mathbb{T}} = \mathbb{L}_T\mathbb{T}$ satisfies*

   1. $\Gamma_\mathbb{T} = \Gamma_{\overline{\mathbb{T}}}$.

   2. If U is the element of minimal charge in $\mathbb{T}$, then $\mathbb{L}_TU$ is the minimal element in $\overline{\mathbb{T}}$.

In particular, if we assume that $\mathbb{A}_{T'}^{(k)} = \mathbb{B}_{(\ell^{k-\ell+1})}\mathbb{A}_T^{(k)}$, then $\mathbb{A}_{T'}^{(k)} \subseteq \mathbb{H}_{\mu|_{(\ell^h)}}$ and by Conjecture 10, the poset $(\mathbb{A}_{T'}^{(k)}, <_{cc})$ is connected. Therefore, for any tableau U of shape $(\ell^{k-\ell+1})$, $\mathbb{L}_U\mathbb{A}_{T'}^{(k)}$ satisfies the conditions above. However, these are exactly the conditions on a copy of an atom (see 5.3) and thus $\mathbb{L}_U\mathbb{A}_{T'}^{(k)}$ is a copy.

Corollary 23. *If $\mathbb{A}_{T_1}^{(k)} = \mathbb{B}_{(\ell^h)}\mathbb{A}_{T_2}^{(k)}$ is a copy of $\mathbb{A}_\lambda^{(k)}$, then for each tableau T of shape (ℓ^h),*

$$\mathbb{A}_{\mathbb{L}_T T_1}^{(k)} = \mathbb{L}_T \, \mathbb{A}_{T_1}^{(k)} \tag{6.8}$$

is another copy of $\mathbb{A}_\lambda^{(k)}$.

In example 6.7, we let $111 \to 123$ to obtain another copy of $\mathbb{A}_{3,2,1}^{(3)}$:

$$\mathbb{A}_{546123}^{(3)} = \boxed{\begin{smallmatrix}5\\4\,6\\1\,2\,3\end{smallmatrix}} + \boxed{\begin{smallmatrix}4\\1\,2\,3\,5\,6\end{smallmatrix}} + \boxed{\begin{smallmatrix}6\\4\\1\,2\,3\,5\end{smallmatrix}} + \boxed{\begin{smallmatrix}4\,6\\1\,2\,3\,5\end{smallmatrix}}. \tag{6.9}$$

Proof of Property 22. Let $T_{(\ell^h)}$ be the tableau of shape and evaluation (ℓ^h). Every element $U \in \mathbb{T}$ contains $T_{(\ell^h)}$, and any letter in $U/T_{(\ell^h)}$ is larger than those in $T_{(\ell^h)}$. Therefore the cyclage or cocyclage that links two elements U and U' of $\mathbb{T}$ does not involve the letters in $T_{(\ell^h)}$ and we can thus change the content of this subtableau without affecting the cyclage-cocyclage relations, as long as the new subtableau also contains the smallest letters. Hence, the Hasse diagrams of the posets $(\mathbb{T}, <_{cc})$ and $(\overline{\mathbb{T}}, <_{cc})$ are identical. The second condition follows from the connectedness of the poset $(\mathbb{T}, <_{cc})$ which implies that cyclage-cocyclage relations could not be preserved if the element of minimal charge in $\overline{\mathbb{T}}$ was not $\mathbb{L}_T$ applied on the element of minimal charge of $\mathbb{T}$. $\qquad\square$

Another consequence of Conjecture 21 arises from the case $t = 1$. Here, the action of $\mathbb{B}_{(\ell^h)}$ on a tableau T is associated to the multiplication of the Schur functions $S_{(\ell^h)}S_{\text{shape}(T)}$.

Corollary 24. *Assuming Conjecture 21 holds, if we let $A_\lambda^{(k)} = A_\lambda^{(k)}[X; 1]$, then*

$$S_{(\ell^{k-\ell+1})} \, A_\lambda^{(k)} = A_{\lambda \cup (\ell^{k-\ell+1})}^{(k)}. \tag{6.10}$$

We have now seen that any atom can be understood as the application of rectangular promotion operators to an irreducible component. Our study is thus reduced to examining the irreducibles (atoms of level k that cannot be obtained by applying k-rectangular operators to a smaller atom). Interestingly, we can obtain the level k atom indexed by the irreducible partition of maximal degree,

$$\lambda_M = \left((k-1)^1, (k-2)^2, \cdots, 1^{k-1}\right), \tag{6.11}$$

by a recursive application of $(k-1)$-rectangular promotion operators on the empty tableau.

Conjecture 25. *The maximal irreducible atom of level k is an atom of level $k-1$;*

$$A_{\lambda_M}^{(k)}[X; t] = A_{\lambda_M}^{(k-1)}[X; t]. \tag{6.12}$$

Furthermore, from Conjecture 21, this atom is simply

$$A_{\lambda_M}^{(k)}[X; t] = A_{\lambda_M}^{(k-1)}[X; t] = \mathcal{F}\left(\mathbb{B}_{(k-1)}\mathbb{B}_{((k-2)^2)} \cdots \mathbb{B}_{(1^{k-1})} \, \mathbb{H}_0\right), \tag{6.13}$$

For example, the atom $A^{(3)}_{2,1,1}[X;t]$ is given by

$$A^{(3)}_{2,1,1}[X;t] = F\left(\mathbb{B}_{(2)}\,\mathbb{B}_{(1^2)}\,\mathbb{H}_0\right) = F\left(\begin{array}{c}\boxed{\begin{smallmatrix}3\\2\\1\;1\end{smallmatrix}}\end{array} + \boxed{\begin{smallmatrix}2\\1\;1\;3\end{smallmatrix}}\right) = S_{2,1,1}[X;t] + t\,S_{3,1}[X;t]. \quad (6.14)$$

When $t = 1$, $V_k = \{H_\lambda[X;t]\}_{\lambda_1 \le k}$ reduces to the polynomial ring $\mathbb{Q}[h_1, \ldots, h_k] = V_k(1)$. If $\mathcal{I}_k$ denotes the ideal generated by the k-rectangular Schur functions $S_{(\ell^{k+1-\ell})}$, we have the following proposition.

Proposition 26. *The homogeneous functions indexed by k-irreducible partitions form a basis of the quotient ring $V_k(1)/\mathcal{I}_k$.*

Proof.   For a partition λ bounded by k, we set

$$\tilde{h}_\lambda = \begin{cases} h_\lambda & \text{if } \lambda \text{ is } k\text{-irreducible} \\ \tilde{h}_{\mu+(\ell^{k+1-\ell})} = S_{(\ell^{k+1-\ell})}\,\tilde{h}_\mu & \text{if } \lambda = \mu \cup (\ell^{k+1-\ell}) \end{cases}. \quad (6.15)$$

These elements are indexed by k-bounded partitions and thus, if independent, span a space with the same dimension as $V_k(1)$. In fact, the $\tilde{h}_\lambda$ form a basis for $V_k(1)$ since $S_{(\ell^{k+1-\ell})} = \det{(h_{\ell-i+j})}_{1 \le i,j \le k+1-\ell}$ implies that $\tilde{h}_\lambda \in V_k(1)$; and $S_\lambda = h_\lambda + \sum_{\mu > \lambda} c_{\mu\lambda} h_\mu$ gives $\tilde{h}_\lambda = h_\lambda + \sum_{\mu > \lambda} d_{\mu\lambda} h_\mu$, which implies that they are independent.

First note that the $\tilde{h}_\lambda$ span the quotient ring $V_k(1)/\mathcal{I}_k$ because they span $V_k(1)$. Since by definition $\tilde{h}_\mu \equiv 0$ in the quotient ring when μ is not k-irreducible, the $\tilde{h}_\lambda$ indexed by k-irreducible partitions will form a basis for the quotient ring $V_k(1)/\mathcal{I}_k$ if they are independent in $V_k(1)/\mathcal{I}_k$. Let $\mathcal{S}$ be the set of all k-irreducible partitions. If, in $V_k(1)/\mathcal{I}_k$, we have

$$\sum_{\lambda \in \mathcal{S}} d_\lambda\,\tilde{h}_\lambda = 0\,, \quad (6.16)$$

then, in $V_k(1)$, we must have

$$\sum_{\lambda \in \mathcal{S}} d_\lambda\,\tilde{h}_\lambda = \sum_i C_i\,S_{(i^{k+1-i})}\,, \quad (6.17)$$

for some $C_i \in V_k(1)$. Further, since $C_i = \sum_\mu c_{i,\mu}\tilde{h}_\mu$ for some $c_{i,\mu}$, we have

$$\sum_{\lambda \in \mathcal{S}} d_\lambda\,\tilde{h}_\lambda = \sum_{i,\mu} c_{i,\mu}\tilde{h}_\mu S_{(i^{k+1-i})} = \sum_{i,\mu} c_{i,\mu}\tilde{h}_{\mu+(i^{k+1-i})}\,. \quad (6.18)$$

The basis elements appearing in the l.h.s of 6.18 are each indexed by k-irreducible partitions whereas those appearing in the r.h.s are indexed by non-k-irreducible partitions. Therefore, $d_\lambda = 0$ for all λ and by 6.16, this proves that the $\tilde{h}_\lambda$ indexed by k-irreducible partitions are independent in $V_k(1)/\mathcal{I}_k$. $\square$

We now have that the dimension of the quotient $V_k(1)/\mathcal{I}_k$ is $k!$. Since we assume that the atoms of level k form a basis for V_k, Corollary 24 implies that the k-irreducible atoms also form a basis of $V_k(1)/\mathcal{I}_k$, since the atoms generate $V_k(1)/\mathcal{I}_k$, and the only possibly non-zero atoms in $V_k(1)/\mathcal{I}_k$ are the $k!$ irreducible ones.

Corollary 27. *Assuming the atoms of level k form a basis of V_k and Conjecture 21 holds, the k-irreducible atoms form a basis of the quotient ring $V_k(1)/\mathcal{I}_k$.*

If we link all atoms that occur in the action of e_1 on a given atom in $V_k(1)/\mathcal{I}_k$, we obtain a poset illustrated in Figure 5. The rank generating function of this poset was given in 1.26. This poset seems to have a remarkable symmetry property called *flip*-invariance.

Definition 28. *Given a k-irreducible partition of the form*

$$\lambda = \left((k-1)^{n_1}, (k-2)^{n_2}, \cdots, 1^{n_{k-1}}\right) \quad \text{with } n_i \leq i \text{ for all } i, \tag{6.19}$$

the involution called flip $\mathbf{f}^{(\mathbf{k})}$ is defined by

$$\mathbf{f}^{(\mathbf{k})} A_\lambda^{(k)} = A_{\lambda^{\mathbf{f}^{(\mathbf{k})}}}^{(k)} \tag{6.20}$$

where $\quad \lambda^{\mathbf{f}^{(\mathbf{k})}} = \left((k-1)^{1-n_1}, (k-2)^{2-n_2}, \cdots, 1^{k-1-n_{k-1}}\right).$

$$\tag{6.21}$$

For instance,

$$\mathbf{f}^{(\mathbf{5})} A_{4,3,2}^{(5)} = A_{3,2,2,1,1,1,1}^{(5)}.$$

Conjecture 29. *The poset associated to the action of e_1 on atoms in $V_k(1)/\mathcal{I}_k$ is flip-invariant. That is, if there is an arrow between two atoms $A_\mu^{(k)}$ and $A_\lambda^{(k)}$, then there will be an arrow between the two atoms $A_{\mu^{\mathbf{f}^{(\mathbf{k})}}}^{(k)}$ and $A_{\lambda^{\mathbf{f}^{(\mathbf{k})}}}^{(k)}$.*

Given the $k!$ irreducible atoms, from which all other atoms are constructed using k-rectangular promotion operators, the complete decomposition of the standard tableaux into atoms can in principle be obtained. We give here the cases $k = 2$ and $k = 3$.

6.1 Case $k = 2$ and $k = 3$

We start with $k = 2$. If $\mathcal{S}_n$ denotes the set of standard tableaux on n letters, then

$$\left(\overline{\mathbb{B}}_{(2)} + \overline{\mathbb{B}}_{(1^2)}\right) \mathcal{S}_n = \mathcal{S}_{n+2}, \tag{6.22}$$

where $\overline{\mathbb{B}}_{(2)} = \mathbb{L}_{\boxed{1\,2}} \mathbb{B}_{(2)}$ and $\overline{\mathbb{B}}_{(1^2)} = \mathbb{B}_{(1^2)}$. This recursion implies, for $\mathbb{A}_0^{(2)} = \mathbb{H}_0$ and $\mathbb{A}_1^{(2)} = \boxed{1}$,

$$\left(\overline{\mathbb{B}}_{(2)} + \overline{\mathbb{B}}_{(1^2)}\right)^\ell \mathbb{A}_\epsilon^{(2)} = \mathcal{S}_{2\ell+\epsilon}, \quad \text{where } \epsilon \in \{0, 1\}. \tag{6.23}$$

Expanding the left hand side gives

$$\sum_{(v_1,\ldots,v_m)} \overline{\mathbb{B}}_{(v_1^{3-v_1})} \cdots \overline{\mathbb{B}}_{(v_m^{3-v_m})} \mathbb{A}_\epsilon^{(2)} = \mathcal{S}_{2\ell+\epsilon}, \quad \text{for } v_i \in \{1,2\}, \tag{6.24}$$

and each of the standard tableaux must occur in exactly one term of this sum. This is, each standard tableau must occur in exactly one family, denoted

$$\mathbb{A}_{(v_1,\ldots,v_m,\epsilon)}^{(2)} = \overline{\mathbb{B}}_{(v_1^{3-v_1})} \cdots \overline{\mathbb{B}}_{(v_m^{3-v_m})} \mathbb{A}_\epsilon^{(2)}, \qquad v_i \in \{1,2\}. \tag{6.25}$$

We have thus decomposed the set of standard tableaux into these families, which are the atoms of level 2 by Conjecture 21. Furthermore, from Conjecture 20, given a standard tableau, we can determine to which family $\mathbb{A}_{(v_1,\ldots,v_m,\epsilon)}^{(2)}$ belongs, by first performing a (2)-katabolism ($v_1 = 2$) if it contains the subword (12) and otherwise a $(1,1)$-katabolism ($v_1 = 1$). Repeating this procedure on the resulting tableau (until there is only one box left ($\epsilon = 1$) or no boxes left ($\epsilon = 0$)), we obtain the sequence $(v_1,\ldots,v_m)$ that we need.

Now, from Conjecture 21,

$$\mathcal{F}\left(\mathbb{A}_{(v_1,\ldots,v_m,\epsilon)}^{(2)}\right) = t^* A_\lambda^{(2)}[X;t], \tag{6.26}$$

where λ is the partition rearrangement of $(v_1^{3-v_1},\ldots,v_m^{3-v_m},\epsilon)$ and $*$ is a power of t. The symmetric function analogues of $\mathbb{B}_{(2)}$ and $\mathbb{B}_{(1^2)}$ are the vertex operators $B_{(2)}$ and $B_{(1^2)}$ (see next subsection). Therefore, Conjecture 21 suggests that

$$B_{(v_1^{3-v_1})} \cdots B_{(v_m^{3-v_m})} A_\epsilon^{(2)}[X;t] = t^* A_\lambda^{(2)}[X;t], \qquad \epsilon = 0,1, \tag{6.27}$$

where λ is the partition rearrangement of $(v_1^{3-v_1},\ldots,v_m^{3-v_m},\epsilon)$ and $*$ is a power of t. This conjecture connects the atoms to the Macdonald polynomials, since the creation operators that build the Macdonald polynomials recursively can be divided into the operators $B_{(2)}$ and $B_{(1^2)}$ [7, 21]. The positive expansion of Macdonald polynomials indexed by 2-bounded partitions (equivalently, partitions with $\ell(\lambda) \leq 2$) into atoms of level 2 is thus conjecturally the one given in [7, 21] (and to [18], since [19] proves the operators are related to the functions studied in [18]).

In the case $k = 3$, we have the 8 irreducible atoms of 6 distinct shapes,

$$\mathbb{A}_0^{(3)} = \mathbb{H}_0 ; \quad \mathbb{A}_1^{(3)} = \boxed{1}; \quad \mathbb{A}_{12}^{(3)} = \boxed{1\,2}; \quad \mathbb{A}_{21}^{(3)} = \boxed{\begin{smallmatrix}2\\1\end{smallmatrix}};$$

$$\mathbb{A}_{312}^{(3)} = \boxed{\begin{smallmatrix}3\\1\,2\end{smallmatrix}}; \quad \mathbb{A}_{213}^{(3)} = \boxed{\begin{smallmatrix}2\\1\,3\end{smallmatrix}}; \quad \mathbb{A}_{4312}^{(3)} = \boxed{\begin{smallmatrix}4\\3\\1\,2\end{smallmatrix}} + \boxed{\begin{smallmatrix}3\\1\,2\,4\end{smallmatrix}}; \quad \mathbb{A}_{4213}^{(3)} = \boxed{\begin{smallmatrix}4\\2\\1\,3\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\1\,3\,4\end{smallmatrix}}, \tag{6.28}$$

from which we can build any atom of evaluation $(1,\ldots,1)$ using the promotion operators:

$$\mathbb{B}_{\boxed{1\,2\,3}}, \mathbb{B}_{\boxed{1\,2\,4}}, \mathbb{B}_{\boxed{1\,3\,4}}, \mathbb{B}_{\boxed{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}}, \mathbb{B}_{\boxed{\begin{smallmatrix}4\\2\\1\end{smallmatrix}}}, \mathbb{B}_{\boxed{\begin{smallmatrix}4\\3\\1\end{smallmatrix}}}, \mathbb{B}_{\boxed{\begin{smallmatrix}4\\1\,2\end{smallmatrix}}}, \mathbb{B}_{\boxed{\begin{smallmatrix}2\,4\\1\,3\end{smallmatrix}}}, \mathbb{B}_{\boxed{\begin{smallmatrix}3\,5\\1\,2\end{smallmatrix}}}, \mathbb{B}_{\boxed{\begin{smallmatrix}2\,5\\1\,3\end{smallmatrix}}}. \tag{6.29}$$

Here an operator indexed by a tableau T of shape R is $\mathbb{L}_T \mathbb{B}_R$ followed by the reindexation of the letters not in T such that the resulting tableaux are standard. For instance,

$$\mathbb{B}_{\boxed{1\,3\,4}} \; \boxed{\begin{smallmatrix}2\\1\end{smallmatrix}} = \boxed{\begin{smallmatrix}5\\2\\1\,3\,4\end{smallmatrix}} + \boxed{\begin{smallmatrix}2\\1\,3\,4\,5\end{smallmatrix}}. \tag{6.30}$$

Using 6.28 and 6.29, we consider the sets of tableau

$$\mathbb{A}^{(3)}_{(T_1,\ldots,T_m,T)} = \mathbb{B}_{T_1} \cdots \mathbb{B}_{T_m} \, \mathbb{A}^{(3)}_T \,, \tag{6.31}$$

for sequences $(T_1,\ldots,T_m,T)$ that obey the following rules (read from right to left):

1. $\boxed{\begin{smallmatrix}3\,5\\1\,2\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}2\,5\\1\,3\end{smallmatrix}}$ can only follow a tableau that contains the subtableau $\boxed{1}$.

2. $\boxed{1\,2\,4}$ and $\boxed{1\,3\,4}$ can only follow a tableau that contains the subtableau $\boxed{\begin{smallmatrix}2\\1\end{smallmatrix}}$.

3. $\boxed{\begin{smallmatrix}4\\2\\1\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}4\\3\\1\end{smallmatrix}}$ can only follow a tableau that contains the subtableau $\boxed{1\,2}$.

We conjecture that there is a one-to-one correspondence between the sequences $(T_1,\ldots,T_m,T)$, and the set of tableaux indexing all level 3 copy atoms with standard evaluation. Moreover, we can determine to which atom an arbitrary standard tableaux belongs in view of Conjecture 20; katabolism is the inverse of rectangular promotion. That is, given a tableau U, we can determine which sequence $(T_1,\ldots,T_m,T)$ can be extracted by katabolism from U.

6.2 Generalized Kostka polynomials

Given a sequence of partitions $S = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$, the generalized Kostka polynomial $H_S[X;t]$ found in [18] is a t-generalization of the product of Schur functions indexed by the partitions in S (different approaches to these polynomials include those in [8, 15]). More precisely, if we consider only its term of degree $n = |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(m)}|$,

$$H_S[X;t] = \sum_{\lambda \vdash n} K_{\lambda;S}(t) \, S_\lambda[X] \,, \tag{6.32}$$

where, for the scalar product $\langle\,,\,\rangle$ on which the Schur functions are orthonormal,

$$K_{\lambda;S}(1) = \langle S_\lambda[X], S_{\lambda^{(1)}}[X] \, S_{\lambda^{(2)}}[X] \cdots \rangle \,. \tag{6.33}$$

If successively reading the entries of $\lambda^{(1)}$, $\lambda^{(2)}, \ldots$ produces a partition μ, S is said to be dominant. In this case, it has been conjectured [18] that

$$H_S[X;t] = \sum_{T \in \mathbb{H}_S} t^{\text{charge}(T)} \, S_{\text{shape}(T)}[X] \,, \tag{6.34}$$

where $\mathbb{H}_S$ is the set of tableaux T of evaluation μ such that $\mathbb{P}_S(T) = T$ (see section 3). Now, if $S = \big((\ell_1^{k+1-\ell_1}), \dots, (\ell_m^{k+1-\ell_m})\big)$ is a dominant sequence of k-rectangles, then Conjecture 21 implies that for $\mu = \big(\ell_1^{k+1-\ell_1}, \dots, \ell_m^{k+1-\ell_m}\big)$,

$$\mathbb{B}_{(\ell_1^{k+1-\ell_1})} \cdots \mathbb{B}_{(\ell_m^{k+1-\ell_m})} \mathbb{H}_0 = \mathbb{A}_\mu^{(k)}. \tag{6.35}$$

Moreover, by the definition of atoms we have that $\mathbb{P}_S(\mathbb{A}_\mu^{(k)}) = \mathbb{A}_\mu^{(k)}$ since $\mu^{\to k} = S$. Therefore, $\mathbb{H}_S = \mathbb{A}_\mu^{(k)}$ since both sets contain the same number of elements (the number of terms in the product of the Schur functions corresponding to shapes $(\ell_1^{k+1-\ell_1}), \dots, (\ell_m^{k+1-\ell_m})$). We thus have the following connection between atoms and the generalized Kostka polynomials:

Conjecture 30. *If $S = \big((\ell_1^{k+1-\ell_1}), \dots, (\ell_m^{k+1-\ell_m})\big)$ is such that $\big(\ell_1^{k+1-\ell_1}, \dots, \ell_m^{k+1-\ell_m}\big)$ is a partition μ, then*

$$A_\mu^{(k)}[X;t] = H_S[X;t]. \tag{6.36}$$

Further, it is shown in [19] that the generalized Kostka polynomials can be defined as

$$H_S[X;t] = B_{\lambda^{(1)}} B_{\lambda^{(2)}} \cdots B_{\lambda^{(m)}} \cdot 1, \tag{6.37}$$

where B_λ corresponds to H_λ^t in their notation. Given our formula 6.35, it is natural to assume that the vertex operators $B_{(\ell^{k+1-\ell})}$ indexed by k-rectangular partitions are the operators that extend Conjecture 21 to the level of symmetric functions.

Conjecture 31. *Given a k-rectangular partition $(\ell^{k+1-\ell})$, we have*

$$B_{(\ell^{k+1-\ell})} A_\lambda^{(k)}[X;t] = t^c A_{\lambda \cup (\ell^{k+1-\ell})}^{(k)}[X;t], \quad \text{where } c \in \mathbb{N}. \tag{6.38}$$

7   The k-conjugation of a partition

Here we introduce a generalization of partition conjugation, defined for partitions bounded by k. When k is large, our k-conjugation reduces to the usual conjugation.

A skew diagram D is said to have hook-lengths bounded by k if the hook-length of any cell in D is not larger than k. For a positive integer $m \le k$, the k-multiplication $m \times^{(k)} D$ is the skew diagram $\overline{D}$ obtained by adding a first column of length m to D such that the number of parts of $\overline{D}$ is as small as possible while ensuring that its hook-lengths are bounded by k. For example,

$$\tag{7.1}$$

Definition 32. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a k-bounded partition and let D be the skew diagram obtained by k-multiplying from right to left the entries of λ:*

$$D = \lambda_1 \times^{(k)} \cdots \times^{(k)} \lambda_n. \tag{7.2}$$

The k-conjugate of λ, denoted λ^{ω_k}, is the vector obtained by reading the number of boxes in each row of D.

When $k \to \infty$, $\lambda^{\omega_k} = \lambda'$ since each k-multiplication step reduces to adding a column of length λ_i at the bottom row.

Property 33. *If λ is a k-bounded partition, then λ^{ω_k} is also a k-bounded partition.*

Proof. λ^{ω_k} is k-bounded since D has hook-lengths bounded by k. To see that λ^{ω_k} is a partition, assume by induction that the parts of $D^{(2)} = \lambda_2 \times^{(k)} \cdots \times^{(k)} \lambda_n$ form a partition μ. The skew diagram $D = \lambda_1 \times^{(k)} D^{(2)}$ is obtained by adding a column of length λ_1 to $D^{(2)}$ starting at some row h. To see that D must also have parts of weakly decreasing size, it suffices to show that $\mu_{h-1} > \mu_h$. Suppose $\mu_{h-1} = \mu_h$ and consider the two possible cases (Figure 3). Keep in mind that any column can be no longer than those to its left since $\lambda_i \leq \lambda_j \,\forall i > j$. If row $h-1$ lies directly below row h, then sliding the new column down to row $h-1$ gives a skew diagram of length less than D with hook-lengths at most k. Therefore our column would not have been added to row h. Now if row $h-1$ lies below and to the right of row h, the column indicated by an arrow can be moved down without producing any hook-lengths longer than k. Since $D^{(2)} = \lambda_2 \times^{(k)} \cdots \times^{(k)} \lambda_n$, this is a contradiction.$\square$
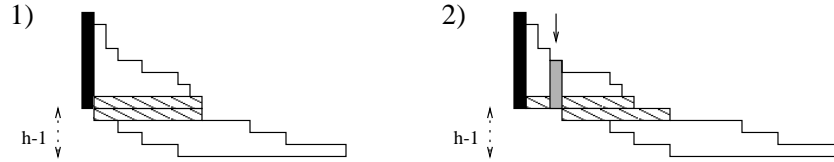

Figure 3:

For example, we can compute $(2, 2, 1, 1)^{\omega_4} = (3, 2, 1)$ by the following steps:

$$\square \times^{(4)} \square \times^{(4)} \square \times^{(4)} \square = \square \times^{(4)} \square \times^{(4)} \square\square = \square \times^{(4)} \square\square\square = \square\square\square. \tag{7.3}$$

Property 34. *For a k-bounded partition λ, let $D = \lambda_1 \times^{(k)} \cdots \times^{(k)} \lambda_n$ and $\overline{D}$ be the skew diagram obtained by shifting any row in D to the left. If the number of columns of $\overline{D}$ is not more than the number of columns of D then the hook-lengths of $\overline{D}$ are not k-bounded.*

Proof. Assume by induction that $D^{(2)} = \lambda_2 \times^{(k)} \cdots \times^{(k)} \lambda_n$, with rows of length μ, satisfies this property. The skew diagram $D = \lambda_1 \times^{(k)} D^{(2)}$ falls into one of the two generic
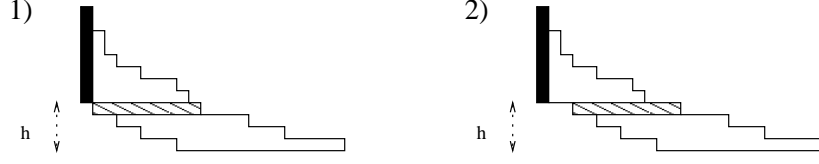
Figure 4:

cases illustrated in Figure 4. In the first case, since the column is added above row h, we know $\lambda_1 + \mu_h > k$. Thus, row h cannot be moved left or we would have a cell with hook-length $\lambda_1 + \mu_h > k$. In the second case, row h cannot be moved left without violating the assumption that $D^{(2)}$ obeys the property. □

Theorem 35. *ω_k is an involution on partitions bounded by k. That is, for λ with $\lambda_1 \leq k$,*

$$(\lambda^{\omega_k})^{\omega_k} = \lambda. \tag{7.4}$$

Proof.   Let $D = \lambda_1 \times^{(k)} \cdots \times^{(k)} \lambda_n$. Property 34 implies that D is recovered by performing the k-multiplication of the entries of λ^{ω_k} in a conjugate way (adding rows to the leftmost position such that the hook-lengths are never larger than k). Therefore, if $\lambda^{\omega_k} = \mu$, the conjugate of D is given by $D' = \mu_1 \times^{(k)} \cdots \times^{(k)} \mu_m$, and thus $(D')' = D$ implies that $\mu^{\omega_k} = (\lambda^{\omega_k})^{\omega_k} = \lambda$. □

Given the k-conjugation of a partition, it is natural to consider the relation among an atom indexed by λ and the atom indexed by λ^{ω_k}. In fact, our examples suggest that conjugating each tableaux in an atom produces the tableaux in another atom.

Conjecture 36. *Let T be a standard tableau. For any copy $\mathbb{A}_T^{(k)}$ of $\mathbb{A}_\lambda^{(k)}$,*

$$\left(\mathbb{A}_T^{(k)}\right)^t = \mathbb{A}_{T'}^{(k)}, \tag{7.5}$$

for some standard tableau T' of shape λ^{ω_k}.

Since, at any level, there is at least one copy of each atom of a given degree in the set of standard tableaux, we have the following corollary:

Corollary 37. *In any atom of shape λ and level k, there is a unique element of maximal charge whose shape is the conjugate of λ^{ω_k}, $(\lambda^{\omega_k})'$.*

Furthermore, since a standard tableau T in n letters satisfies $\mathrm{charge}(T^t) = \binom{n}{2} - \mathrm{charge}(T)$,

Corollary 38. *Let ω be the involution such that $\omega S_\lambda[X] = S_{\lambda'}[X]$. Then, for some $* \in \mathbb{N}$,*

$$\omega\, A_\lambda^{(k)}[X;t] = t^* A_{\lambda^{\omega_k}}^{(k)}[X;1/t]. \tag{7.6}$$

Here we see that for large k, $\lambda^{\omega_k} = \lambda'$ is consistent with the fact that $A_\lambda^{(k)}[X;t] = S_\lambda[X]$ in this case.

8 Pieri rules

Beautiful combinatorial algorithms are known for the Littlewood-Richardson coefficients that appear in a product of Schur functions;

$$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu \, . \tag{8.1}$$

Recall by Property 6 that our atoms $A^{(k)}_\lambda[X;t]$ are simply the Schur functions S_λ when k is large. Therefore the expansion coefficients in a product of atoms are the Littlewood-Richardson coefficients when k is large and it is natural to examine the coefficients in a product of two atoms for general k. In fact, in the case $t = 1$, the coefficients in a product of two atoms do seem to generalize Littlewood-Richardson coefficients.

Conjecture 39. *Let $A^{(k)}_\lambda$ denote the case $t = 1$ in $A^{(k)}_\lambda[X;t]$. Then*

$$A^{(k)}_\lambda A^{(k)}_\mu = \sum_\nu c^{\nu\,(k)}_{\lambda\mu} A^{(k)}_\nu \, , \quad \text{where } 0 \subseteq c^{\nu\,(k)}_{\lambda\mu} \subseteq c^\nu_{\lambda\mu} \, . \tag{8.2}$$

In particular, we know

$$c^{\nu\,(k)}_{\lambda\mu} = c^\nu_{\lambda\mu} \quad \text{for } k \geq |\mu| \, . \tag{8.3}$$

Identity 8.1 reduces to the Pieri rule when λ is a row (resp. column). Since an atom $A^{(k)}_\lambda$ reduces to h_ℓ (resp. e_ℓ) when λ is a row (column) of length $\ell \leq k$, our conjecture can be reduced to a k-generalization of the Pieri rule.

Corollary 40. *For certain sets of shapes $E^{(k)}_{\lambda,\ell}$ and $\bar{E}^{(k)}_{\lambda,\ell}$, we have for $\ell \leq k$,*

$$h_\ell A^{(k)}_\lambda = \sum_{\mu\in E^{(k)}_{\lambda,\ell}} A^{(k)}_\mu \quad \text{and} \quad e_\ell A^{(k)}_\lambda = \sum_{\mu\in \bar{E}^{(k)}_{\lambda,\ell}} A^{(k)}_\mu \, . \tag{8.4}$$

We conjecture the sets $E^{(k)}_{\lambda,\ell}$ and $\bar{E}^{(k)}_{\lambda,\ell}$ can be defined in a manner analogous to the Pieri rule.

Conjecture 41. *For any positive integer $\ell \leq k$,*

$$E^{(k)}_{\lambda,\ell} = \{\mu \,|\, \mu/\lambda \text{ is a horizontal } \ell\text{-strip and } \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a vertical } \ell\text{-strip}\} \, ,$$
$$\bar{E}^{(k)}_{\lambda,\ell} = \{\mu \,|\, \mu/\lambda \text{ is a vertical } \ell\text{-strip and } \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a horizontal } \ell\text{-strip}\} \, . \tag{8.5}$$

For example, to obtain the indices of the elements that occur in $e_2 A^{(4)}_{3,2,1}$, we compute $(3,2,1)^{\omega_4} = (2,2,1,1)$ by Definition 32 and then add a horizontal 2-strip to $(2,2,1,1)$ in all possible ways. This gives $(2,2,2,1,1),(3,2,1,1,1),(3,2,2,1)$ and $(4,2,1,1)$ of which all are 4-bounded. Our set then consists of all the 4-conjugates of these partitions that leave a vertical 2-strip when $(3,2,1)$ is extracted from them. The corresponding 4-conjugates are

$$(2,2,2,1,1)^{\omega_4} = \; , \; (3,2,1,1,1)^{\omega_4} = \; , \; (3,2,2,1)^{\omega_4} = \; , \; (4,2,1,1)^{\omega_4} = \; , \tag{8.6}$$

and of these partitions, only the first three are such that a vertical 2-strip remains when $(3,2,1)$ is extracted. Therefore

$$e_2 A^{(4)}_{3,2,1} = A^{(4)}_{3,3,2} + A^{(4)}_{3,2,2,1} + A^{(4)}_{3,2,1,1,1}, \tag{8.7}$$

which is in fact correct.

9 Hook case

We are able to explicitly determine the functions $A^{(k)}_\lambda[X;t]$ in the case that λ is a hook partition and also to derive properties of atoms indexed by partitions slightly more general than hooks. These results rely on the following property of a row-shaped katabolism.

Property 42. *If T has shape $\lambda = (m,1^r)$ (a hook), then*

$$\mathbb{K}_{(n)} : T \longrightarrow \begin{cases} \bar{T} & if \;\; n \le m \\ 0 & otherwise \end{cases}, \tag{9.1}$$

where $\bar{T}$ is also hook-shaped.

Proof. Consider a tableau T of shape $\lambda = (m,1^r)$. If $n > m$ then T does not contain a row of length n and thus $\mathbb{K}_{(n)}T = 0$. Assume $n \le m$. Let U be the tableau of shape (1^r) obtained by deleting the bottom row of T. By the definition of katabolism, the action of $\mathbb{K}_{(n)}$ on T amounts to row inserting a sequence of strictly decreasing letters (those of U) into a sequence of weakly increasing letters (the last $m-n$ letters in the bottom row of T). The insertion algorithm implies [1] that in this case, no two elements may be added to the same row and therefore, we obtain a hook shape. $\square$

This property leads to the hook content of any atom that is not indexed by a k-generalized hook partition, that is, a partition of the form $(k,\dots,k,\rho_1,\rho_2,\dots)$ for a hook shape $(\rho_1,\rho_2,\dots)$.

Property 43. *If T is a tableau of shape λ, where λ is not a k-generalized hook, then $\mathbb{A}^{(k)}_T$ does not contain any tableaux with a hook shape.*

Corollary 44. *If λ is a partition that is not a k-generalized hook, then*

$$A_\lambda^{(k)}[X;t] = S_\lambda[X] + \sum_{\mu > \lambda} v_{\mu\lambda}^{(k)}(t)\, S_\mu[X]\,, \tag{9.2}$$

where $v_{\mu\lambda}^{(k)}(t) = 0$ for all hook partitions μ.

Proof.   Let $\lambda^{\to k} = (\lambda^{(1)}, \lambda^{(2)}, \dots)$. The condition on λ implies that λ_2 is at least 2. If we first consider such partitions with $\lambda_1 \neq k$ then $\lambda^{(1)}$ cannot be a hook ($\lambda_1 \neq k$ implies that the first partition in the k-split contains at least the first two parts of λ). But if $\lambda^{(1)}$ is not a hook, then any hook-shaped tableau T in $\mathbb{B}_{(\lambda_1)}\left(\mathbb{A}_{(\lambda_2,\lambda_3,\dots)}^{(k)}\right)$ will not contain the shape $\lambda^{(1)}$ and will therefore be sent to zero under $\mathbb{P}_{\lambda \to k}$. On the other hand, if $\lambda_1 = k$ then $\lambda^{(1)} = (k)$. Now any hook-shaped tableau T in $\mathbb{B}_{(\lambda_1)}\left(\mathbb{A}_{(\lambda_2,\lambda_3,\dots)}^{(k)}\right)$ will be sent to a hook under the (k)-katabolism by Property 42. Our claim thus follows recursively on the remaining terms of the k-split of λ. $\square$

If an atom *is* indexed by a k-generalized hook, we can determine it explicitly.

Property 45. *Let $\lambda = (m, 1^r)$ be a k-irreducible hook partition. Then*

$$\mathbb{A}_\lambda^{(k)} = \begin{cases} (r+1)\, r \cdots 2\, 1^m & \text{if } r + m \leq k \\ (r+1)\, r \cdots 2\, 1^m + r \cdots 2\, 1^m\, (r+1) & \text{otherwise} \end{cases}. \tag{9.3}$$

Note, here an element $(r+1)\, r \cdots 2\, 1^m$ denotes the word $(r+1)\, r \cdots 2\, 1\, 1 \cdots 1$.

Proof.   Since $r, m \leq k - 1$ in any k-irreducible partition $\lambda = (m, 1^r)$, we have that $(1^i)^{\to k} = (1^i)$ for $1 \leq i \leq r$. Therefore, on a tableau T with i boxes, $\mathbb{P}_{(1^i) \to k} T \neq 0$ only for T of shape (1^i) and thus

$$\mathbb{A}_{1^r} = \mathbb{P}_{(1^r) \to k}\, \mathbb{B}_1 \cdots \mathbb{P}_{(1^2) \to k}\, \mathbb{B}_1 \mathbb{P}_{(1) \to k}\, \mathbb{B}_1\, \mathbb{H}_0 = r\, r - 1 \cdots 1\,. \tag{9.4}$$

Moreover, it develops that $\mathbb{B}_m\, \mathbb{A}_{1^r} = (r+1)\, r \cdots 2\, 1^m + r \cdots 2\, 1^m\, (r+1)$. Now we have

$$\mathbb{A}_{(m,1^r)} = \mathbb{P}_{(m,1^r) \to k}\left((r+1)\, r \cdots 2\, 1^m + r \cdots 2\, 1^m\, (r+1)\right). \tag{9.5}$$

Since $r + m - k \leq k$, the k-split of $(m, 1^r)$ is

$$(m, 1^r)^{\to k} = \begin{cases} ((m, 1^r)) & \text{if } r + m \leq k \\ ((m, 1^{k-m}), (1^{r+m-k})) & \text{otherwise} \end{cases}. \tag{9.6}$$

In either case, $\mathbb{P}_{(m,1^r) \to k}\left((r+1)\, r \cdots 2\, 1^m\right) = (r+1)\, r \cdots 2\, 1^m$, but since $r \cdots 2\, 1^m\, (r+1)$ never contains shape $(m, 1^r)$, $\mathbb{P}_{(m,1^r) \to k}\left(r \cdots 2\, 1^m\, (r+1)\right) \neq 0$ only in the second case. $\square$

Now by Conjecture 21, we use the given atoms of level k indexed by a k-irreducible hook shape to obtain more general cases including those indexed by a k-generalized hook shape.

Corollary 46. *Assume Conjecture 21 holds. For a sequence of k-rectangles $(R_1, R_2, \dots, R_j)$, let λ be the partition rearrangement of $(R_1, R_2, \dots, R_j, m, 1^r)$. Then*

$$A_\lambda^{(k)}[X;t] \propto F\left(\mathbb{B}_{R_1} \cdots \mathbb{B}_{R_j} \mathbb{A}_{(m,1^r)}\right). \tag{9.7}$$

References

[1] W. Fulton, *Young Tableaux: with Applications to Representation Theory and Geometry*, Cambridge University Press, 1997.

[2] A. M. Garsia and M. Haiman, *A graded representation module for Macdonald's polynomials*, Proc. Natl. Acad. Sci. USA **90** (1993), 3607–3610.

[3] A. M. Garsia and C. Procesi, *On certain graded S_n-modules and the q-Kostka polynomials*, Adv. Math. **87** (1992), 82–138.

[4] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math Soc. **14** (2001), 941–1006.

[5] N. Jing, *Vertex operators and Hall-Littlewood symmetric functions*, Adv. Math. **87** (1991), 226–248.

[6] D.E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math **34** (1970), 709–727.

[7] L. Lapointe and J. Morse, *Tableaux statistics for two part Macdonald polynomials*, math.CO/9812001.

[8] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties*, J. Math. Phys. **38** (1997), 1041–1068.

[9] A. Lascoux, B. Leclerc and J.-Y. Thibon, *The Plactic Monoid*, in *Algebraic Combinatorics on Words*, M. Lothaire, Cambridge University Press (to appear), http://www-igm.univ-mlv.fr/˜berstel/Lothaire/.

[10] A. Lascoux and M.-P. Schützenberger, *Sur une conjecture de H.O. Foulkes*, C.R. Acad. Sc. Paris. **294** (1978), 323–324.

[11] A. Lascoux and M.-P. Schützenberger, *Le monoïde plaxique*, Quaderni della Ricerca scientifica **109** (1981), 129–156.

[12] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Clarendon Press, Oxford, 1995.

[13] G. de B. Robinson, *On the representations of the symmetric group*, Amer. J. Math. **60**, (1938), 745–760.

[14] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math. **13** (1961), 179–191.

[15] A. Schilling and S. Warnaar, *Inhomogeneous lattice paths, generalized Kostka-Foulkes polynomials, and A_{n-1}-supernomials*, Comm. Math. Phys. **202** (1999), 359–401.

[16] M. Shimozono, *A cyclage poset structure for Littlewood-Richardson tableaux*, European J. Combin. **22** (2001), 365–393.

[17] M. Shimozono, *Multi-atoms and monotonicity of generalized Kostka polynomials*, European J. Combin. **22** (2001), 395–414.

[18] M. Shimozono and J. Weyman, *Graded characters of modules supported in the closure of a nilpotent conjugacy class*, European J. Combin. **21** (2000), 257–288.

[19] M. Shimozono and M. Zabrocki, *Hall-Littlewood vertex operators and generalized Kostka polynomials*, Adv. Math. **158** (2001), 66–85.

[20] S. Veigneau, *ACE, an Algebraic Combinatorics Environment for the computer algebra system MAPLE*, *Version 3.0*,1998, http://phalanstere.univ-mlv.fr/˜ace/.

[21] M.A. Zabrocki, *A Macdonald vertex operator and standard tableaux statistics for the two-column (q,t)-Kostka coefficients*, Electron. J. Combinat. **5**, R45 (1998).
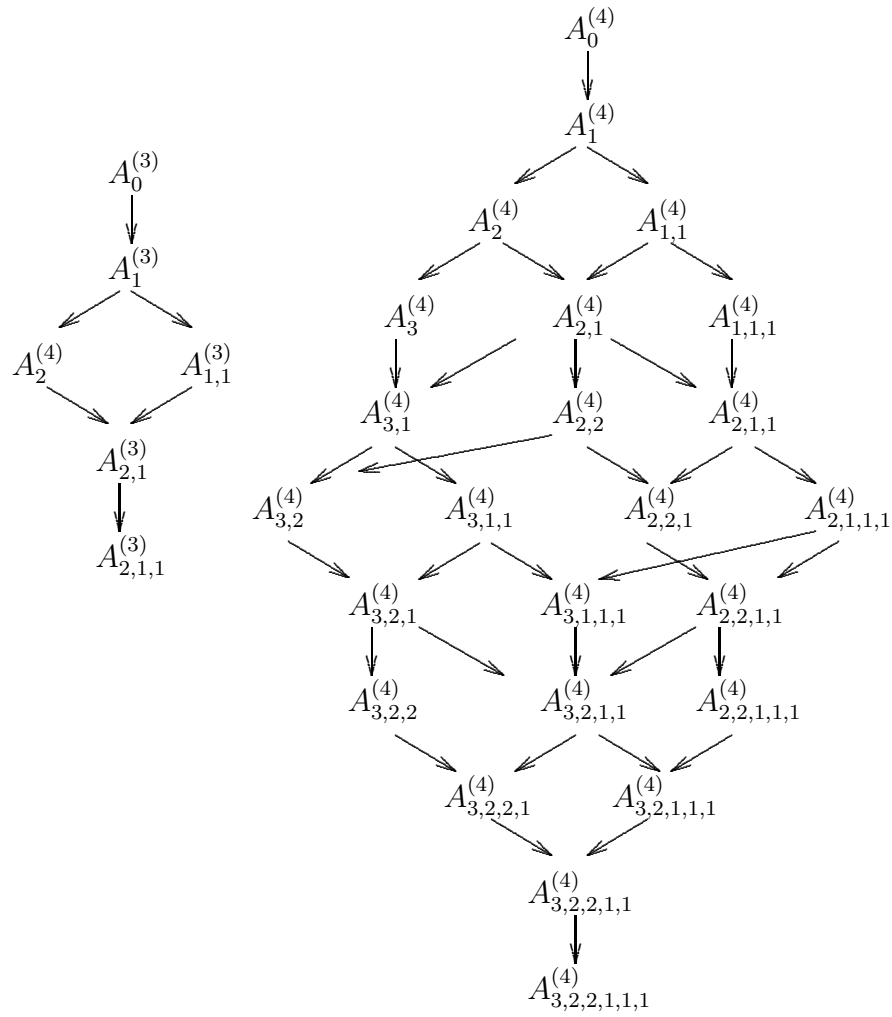
$$A_0^{(4)}$$

$$A_1^{(4)}$$

$$A_0^{(3)}$$

$$A_1^{(3)}$$

$$A_2^{(4)} \qquad A_{1,1}^{(4)}$$

$$A_2^{(4)} \qquad A_{1,1}^{(3)}$$

$$A_3^{(4)} \qquad A_{2,1}^{(4)} \qquad A_{1,1,1}^{(4)}$$

$$A_{2,1}^{(3)}$$

$$A_{3,1}^{(4)} \qquad A_{2,2}^{(4)} \qquad A_{2,1,1}^{(4)}$$

$$A_{2,1,1}^{(3)}$$

$$A_{3,2}^{(4)} \qquad A_{3,1,1}^{(4)} \qquad A_{2,2,1}^{(4)} \qquad A_{2,1,1,1}^{(4)}$$

$$A_{3,2,1}^{(4)} \qquad A_{3,1,1,1}^{(4)} \qquad A_{2,2,1,1}^{(4)}$$

$$A_{3,2,2}^{(4)} \qquad A_{3,2,1,1}^{(4)} \qquad A_{2,2,1,1,1}^{(4)}$$

$$A_{3,2,2,1}^{(4)} \qquad A_{3,2,1,1,1}^{(4)}$$

$$A_{3,2,2,1,1}^{(4)}$$

$$A_{3,2,2,1,1,1}^{(4)}$$

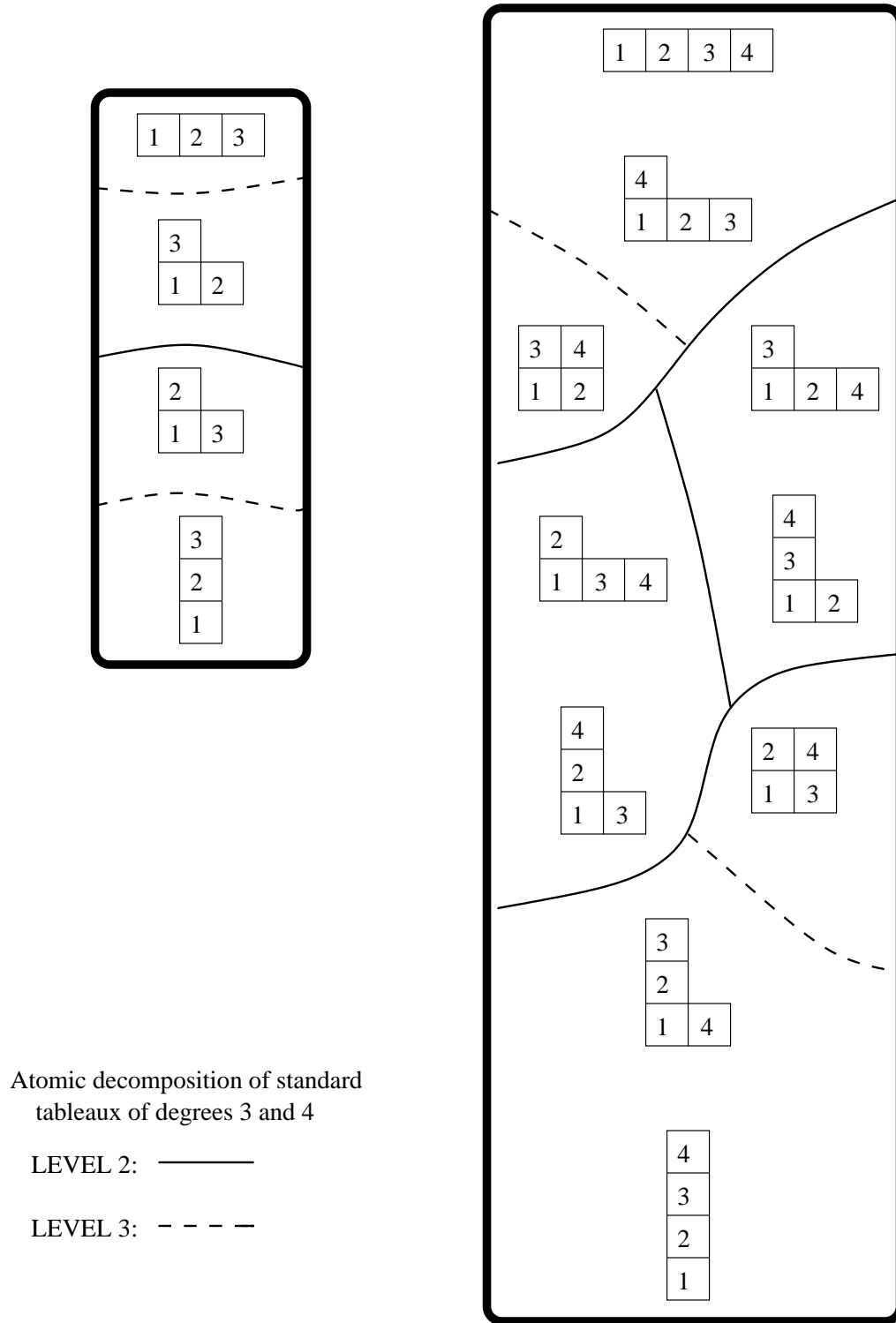Figure 5: Action of e_1 on irreducible atoms of level 3 and 4

Figure 6: In order to read the decomposition of a given level k, one must consider the lines associated to all the levels which are not bigger than k. That is, when doing the decomposition from one level to the other, lines are added without ever being removed. Thus for instance the tableau 2413 and 3214 are in the same atom up to level 2, and in different atoms for any higher levels.

1	2	3	4	5

Atomic decomposition of
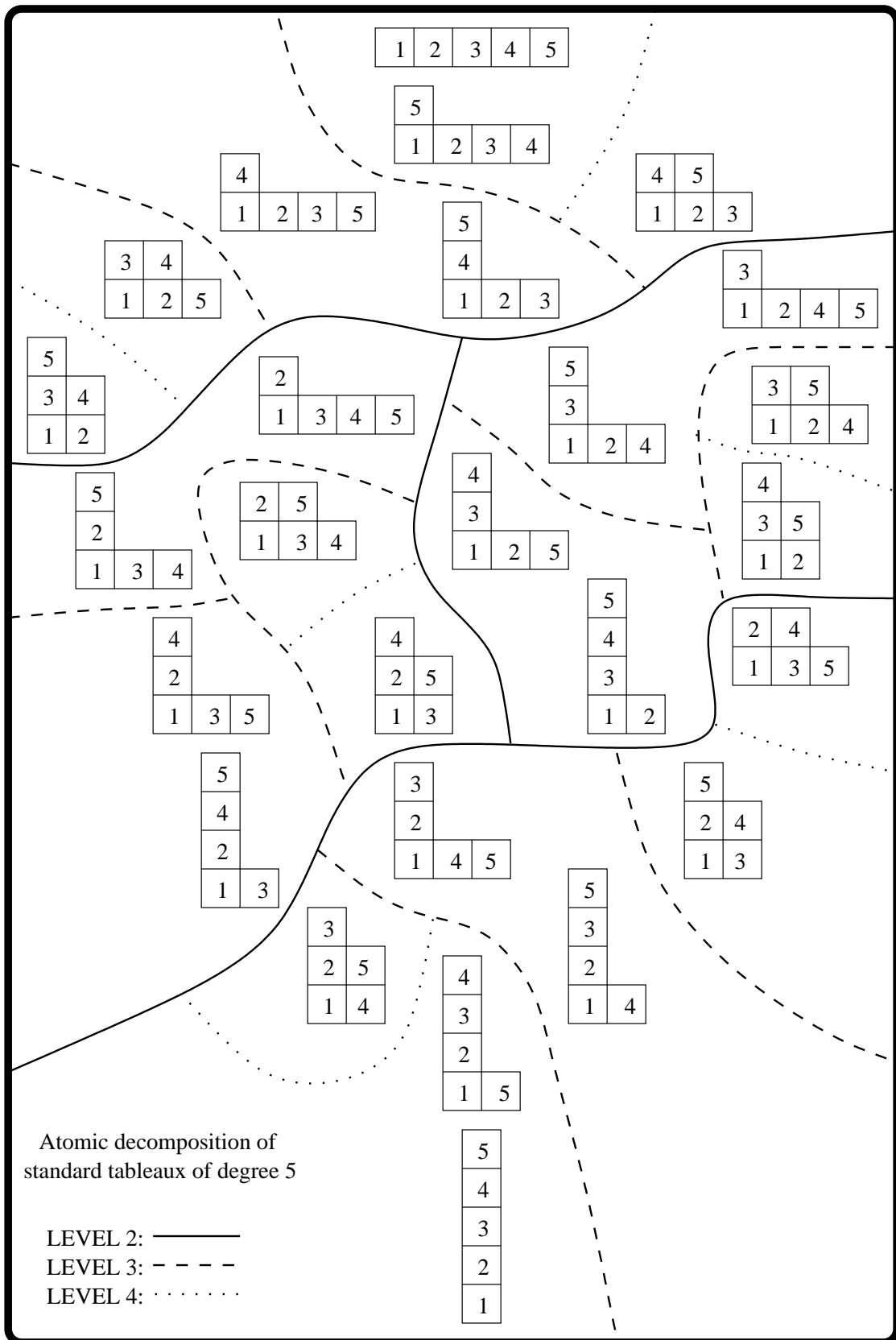standard tableaux of degree 5

LEVEL 2: ———
LEVEL 3: – – – –
LEVEL 4: · · · · · ·

Figure 7: See Figure 6 for details on how to read the figure.